An Improved Neoclassical Drift-Magnetohydrodynamical Fluid Model of Helical Magnetic Island Equilibria in Tokamak Plasmas

Richard Fitzpatrick

Institute for Fusion Studies

Department of Physics

University of Texas at Austin

The effect of the perturbed ion polarization current on the stability of neoclassical tearing modes is calculated using an improved, neoclassical, four-field, drift-MHD model. The calculation involves the self-consistent determination of the pressure and scalar electric potential profiles in the vicinity of the associated magnetic island chain, which allows the chain's propagation velocity to be fixed. Two regimes are considered. First, a regime in which neoclassical ion poloidal flow damping is not strong enough to enhance the magnitude of the polarization current (relative to that found in slab geometry). Second, a regime in which neoclassical ion poloidal flow damping is strong enough to significantly enhance the magnitude of the polarization current. In both regimes, two types of solution are considered. First, a freely rotating solution (i.e., an island chain that is not interacting with a static, resonant, magnetic perturbation). Second, a locked solution (i.e., an island chain that has been brought to rest in the laboratory frame via interaction with a static, resonant, magnetic perturbation). In all cases, the polarization current is found to be either always stabilizing, or stabilizing provided that $\eta_i \equiv d\ln T_i/d\ln n_e$ does not exceed some threshold value. In certain ranges of η_i, the polarization current is found to have have a stabilizing effect on a freely rotating island, but a destabilizing effect on a corresponding locked island.

I. INTRODUCTION

A tokamak is a device that is designed to trap a thermonuclear plasma on a set of toroidally-nested magnetic flux-surfaces.[1] Heat and particles are able to flow around the flux-surfaces relatively rapidly due to the free streaming of charged particles along magnetic field-lines. On the other hand, heat and particles are only able to diffuse across the flux-surfaces relatively slowly, assuming that the magnetic field-strength is large enough to render the particle gyroradii much smaller than the device's minor radius.[2]

Tokamak plasmas are subject to a number of macroscopic instabilities that limit their effectiveness.[3] Such instabilities can be divided into two broad classes. So-called ideal instabilities are non-reconnecting modes that disrupt the plasma in a matter of micro-seconds. However, such instabilities can easily be avoided by limiting the plasma pressure and the net toroidal current.[4] Tearing modes, on the other hand, are relatively slowly growing instabilities that are more difficult to avoid.[4,5] These instabilities tend to saturate at relatively low levels,[6–9] in the process reconnecting magnetic flux-surfaces to form helical structures known as magnetic island chains. Magnetic island chains are radially localized structures centered on so-called rational flux-surfaces, which satisfy $\mathbf{k}\cdot\mathbf{B} = 0$, where $\mathbf{k}$ is the wave-number of the instability, and $\mathbf{B}$ the equilibrium magnetic field. Island chains degrade plasma confinement because they enable heat and particles to flow very rapidly along field-lines from their inner to their outer radii, implying an almost complete loss of confinement in the region lying between these radii.[10]

As is well known, tearing mode dynamics in high-temperature tokamak plasmas is poorly described by the standard, single-fluid, resistive-magnetohydrodynamical (MHD) model.[11] Indeed, in order to obtain realistic predictions, at an absolute minimum, the resistive-MHD model must be replaced by a two-fluid, drift-MHD model. Broadly speaking, the drift-MHD model predicts the existence of two separate branches of nonlinear tearing mode solutions.[12–15] *Ion-branch* solutions are characterized by a flattened pressure profile within the island chain's magnetic separatrix, a relatively large radial island width (compared to the poloidal ion gyroradius), a propagation velocity similar to that of the unperturbed local perpendicular ion fluid velocity, and no emission of drift-waves. On the other hand, *electron-branch* solutions are characterized by a non-flattened pressure profile within the magnetic separatrix, a relatively small radial width, a propagation velocity close to that of the unperturbed local perpendicular electron fluid velocity, and the emission of drift-waves. Numerical simulations suggest that the ion solution branch ceases to exist below a critical island width, whereas the electron solution branch ceases to exist above a second, somewhat larger, critical width.[12,13] The disappearance of one branch of solutions is associated with a bifurcation to the other branch.[12,13]

This paper is concerned with the ion branch of nonlinear tearing mode solutions. The flattening of the pressure profile in the region lying within the island separatrix of such solutions gives rise to the disappearance of the neoclassical bootstrap current [16] there, which has a strong destabilizing effect on the mode.[17] Indeed, this effect is so marked that, unless countered, it would give rise to the formation of magnetic island chains on every rational surface within the plasma, causing the complete destruction of magnetic flux-surfaces.[18] In reality, this is not found to be the case. Instead, so-called *neoclassical tearing modes* (i.e., tearing modes driven unstable by the perturbed bootstrap current) are only observed to form on a few low mode-number rational surfaces within the plasma.[19] This implies the existence of a stabilizing mechanism that counters the destabilizing effect of the perturbed bootstrap current. Two possible mechanisms have been identified in the literature. First, the finite parallel transport in tokamak plasmas, combined with enhanced perpendicular transport due to plasma turbulence, may not allow the flattening of the pressure profile within the magnetic separatrix.[20] However, this mechanism is only effective for relatively thin islands, and is not relevant to the ion solution branch. The second stabilization mechanism, which appears to be the only feasible mechanism for the ion branch, is associated with the perturbed ion polarization current.[21]

Calculating the effect of the perturbed ion polarization current on nonlinear tearing mode stability in a two-fluid plasma turns out to be a rather difficult task, for a number of reasons. The first difficulty is that the sign of the polarization term in the island width evolution equation depends crucially on the island propagation velocity. Generally speaking, the polarization term has one sign if the propagation velocity lies between the unperturbed local perpendicular guiding-center fluid velocity and the unperturbed local perpendicular ion fluid velocity, and the opposite sign otherwise.[21,22] Thus, a meaningful calculation of the polarization term must also be coupled with a calculation of the island propagation velocity. The latter calculation involves a self-consistent determination of the pressure and scalar electric potential profiles in the vicinity of the island chain.[23,24] The second difficulty is that the dominant contribution to the polarization term originates from a boundary layer on the island chain's magnetic separatrix.[25] This contribution is such that the polarization term is stabilizing when the island propagation velocity lies between the unperturbed local perpendicular guiding-center fluid velocity and the unperturbed local perpendicular ion fluid velocity, and destabilizing otherwise.[23] If the contribution of the boundary layer is omitted then the sign of the polarization term is reversed (so that the term is destabilizing when the island propagation velocity lies between the unperturbed local perpendicular guiding-center fluid velocity and the unperturbed local perpendicular ion fluid velocity, and stabilizing otherwise).[21,22] Unfortunately, the contribution of the separatrix boundary layer to the polarization term is a very sensitive function of the thickness of the layer.[26,27] The final difficulty is that the magnitude of the polarization term is profoundly affected by neoclassical ion poloidal flow damping.[28] Indeed, if the damping is sufficiently large then it gives rise to a coupling of the perpendicular and parallel ion flows that acts to significantly enhance the magnitude of the polarization term.[29,30]

Incidentally, because ion-branch magnetic islands are much wider than the poloidal ion gyroradius (and, hence, the ion banana width), it is reasonable to assume that the response of both trapped and passing ions to the perturbed electric and magnetic fields in the vicinity of the island chain can be adequately captured by a fluid model. Of course, such an assumption would not be not reasonable for island chains whose widths are comparable to, or less than, the ion poloidal gyroradius.[31–33]

The aim of this paper is to present a two-fluid calculation of the ion polarization term appearing in the island width evolution equation of a neoclassical tearing mode in a high-temperature tokamak plasma. The calculation is performed using a neoclassical, four-field, drift-MHD model. The model itself was developed, and gradually improved, in Refs. 34–36. The core of the model is a single-helicity version of the well-known four-field model of Hazeltine, Kotschenreuther, and Morrison.[37] The core model is augmented by phenomenological terms representing anomalous cross-field particle and momentum transport due to small-scale plasma turbulence. Finally, the model includes approximate (i.e., flux-surface averaged) expressions for the divergence of the neoclassical ion and electron stress tensors. These expressions allow us to incorporate the bootstrap current, as well as neoclassical ion poloidal and perpendicular flow damping, into the model. Note that perpendicular flow damping, which is due to nonambipolar transport associated with the breaking of toroidal symmetry by the tearing perturbation (and, possibly, by external magnetic perturbations),[39] is often referred to in the literature as "toroidal" flow damping. This name is somewhat misleading, because the damping actually acts on the perpendicular component of the ion fluid velocity.

This paper is organized as follows. The neoclassical, four-field, drift-MHD model that forms the basic of our analysis is introduced in Sect. II. In Sect. III, we calculate the ion polarization term for the case in which the neoclassical ion poloidal flow damping is not large enough to enhance the term's magnitude. In Sect. IV, we calculate the polarization term in the opposite case in which the flow damping is large enough to significantly enhance the term's magnitude. The paper is summarized in Sect. V.

The general form of the calculations outlined in Sects. III and IV is similar to those described in Ref. 35. However, many of the details of the calculations are significantly modified by the improvements in the expressions for the divergences of the neoclassical stress tensors introduced in Ref. 36. These improvements are as follows. First, we have taken into account the fact that the neoclassical velocities towards which the divergences of the neoclassical stress tensors relax the electron and ion velocities are proportional to local electron and ion temperature gradients, respectively, and are, therefore, affected by the modifications to these gradients induced by the presence of the island chain. Second, we have taken into account the fact that that the divergence of the neoclassical ion perpendicular stress tensor generates a force that is primarily directed perpendicular to magnetic field-lines (within a given flux-surface), rather than in the toroidal direction. In addition, we have incorporated magnetic field-line curvature, the bootstrap current, and independent equilibrium number density, electron temperature, and ion temperature gradients into the model.

II. PRELIMINARY ANALYSIS

A. Fundamental Definitions

Consider a large aspect-ratio, low-β, circular cross-section, tokamak plasma equilibrium of major radius R_0, and toroidal magnetic field-strength B_0. Let us adopt a right-handed, quasi-cylindrical, toroidal coordinate system (r, θ, φ), whose symmetry axis ($r = 0$) coincides with the magnetic axis. The coordinate r also serves as a label for the unperturbed (by the island chain) magnetic flux-surfaces. Let the equilibrium toroidal magnetic field and toroidal plasma current both run in the $+\varphi$ direction.

Suppose that a helical magnetic island chain, with m_θ poloidal periods, and n_φ toroidal periods, is embedded in the aforementioned plasma. The island chain is assumed to be radially localized in the vicinity of its associated rational surface, minor radius r_s, which is defined as the unperturbed magnetic flux-surface at which $q(r_s) = m_\theta/n_\varphi$. Here, $q(r)$ is the safety-factor profile (which is assumed to be a monotonically increasing function of r). Let the full radial width of the island chain's magnetic separatrix be $4\,w$. In the following, it is assumed that $r_s/R_0 \ll 1$ and $w/r_s \ll 1$.

The plasma is conveniently divided into an inner region, that comprises the plasma in the immediate vicinity of the rational surface (and includes the island chain), and an outer region that comprises the remainder of the plasma. As is well known, in a high-temperature tokamak plasma, linear, ideal, MHD analysis invariably suffices to calculate the mode structure in the outer region, whereas nonlinear, nonideal, drift-MHD analysis is generally required in the inner region. Let us assume that the linear, ideal, MHD solution has been found in the outer region. In the absence of an external perturbation, such a solution is characterized by a single real parameter, Δ', (with units of inverse length) known as the tearing stability index.[5] The tearing stability index measures the free energy available in the outer region to cause a spontaneous change in the island chain's radial width. This free energy acts to increase the width if $\Delta' > 0$, and vice versa. It remains to obtain a nonlinear, nonideal, drift-MHD solution in the inner region, and then to asymptotically match this solution to the aforementioned linear, ideal, MHD solution at the boundary between the inner and outer regions.

All fields in the inner region are assumed to depend only on the normalized radial coordinate $X = (r - r_s)/w$, and the helical angle $\zeta = m_\theta\,\theta - n_\varphi\,\varphi - \phi_p(t)$. In particular, the electron number density, electron temperature, and ion temperature profiles in the inner region take the forms $n(X,\zeta) = n_0\,(1 + \delta n/n_0)$, $T_e(X,\zeta) = T_{e\,0}\,(1 + \eta_e\,\delta n/n_0)$, and $T_i(X,\zeta) = T_{i\,0}\,(1 + \eta_i\,\delta n/n_0)$, respectively, Here, n_0, $T_{e\,0}$, $T_{i\,0}$, η_e, and η_i are uniform constants. Moreover, $\delta n(X,\zeta)/n_0 \to -(w/L_n)\,X$ as $|X| \to \infty$, where $L_n > 0$ is the density scale-length at the rational surface. Note that we are assuming, for the sake of simplicity, that $\delta T_e/T_{e\,0} = \eta_e\,\delta n/n_0$, and $\delta T_i/T_{i\,0} = \eta_i\,\delta n/n_0$, where $\delta T_e = T_e - T_{e\,0}$, et cetera. It follows that the flattening of the electron density profile within the island separatrix also implies the flattening of the electron and ion temperature profiles. This approach is suitable for relatively wide, ion-branch magnetic island chains, where we expect complete flattening of the pressure profile within the island separatrix, but would not be suitable for relatively narrow, electron-branch island chains, where we expect the electron temperature profile to be flattened, but not the electron density and ion temperature profiles.[14,15]

It is convenient to define the poloidal wavenumber, $k_\theta = m_\theta/r_s$, the resonant safety-factor, $q_s = m_\theta/n_\varphi$, the inverse aspect-ratio, $\epsilon_s = r_s/R_0$, the ion diamagnetic speed,

$$V_{*\,i} = \frac{T_{i\,0}\,(1+\eta_i)}{e\,B_0\,L_n}, \tag{1}$$

the electron diamagnetic speed, $V_{*\,e} = \tau\,V_{*\,i}$, where

$$\tau = \left(\frac{T_{e\,0}}{T_{i\,0}}\right)\left(\frac{1+\eta_e}{1+\eta_i}\right), \tag{2}$$

the poloidal ion gyroradius,

$$\rho_{\theta\,i} = \left(\frac{q_s}{\epsilon_s}\right)\left[\frac{T_{i\,0}\,(1+\eta_i)}{m_i}\right]^{1/2}\left(\frac{m_i}{e\,B_0}\right), \tag{3}$$

and the ion beta,

$$\beta_i = \frac{\mu_0\,n_0\,T_{i\,0}\,(1+\eta_i)}{B_0^{\,2}}. \tag{4}$$

All of these quantities are evaluated at the rational surface. Here, e is the magnitude of the electron charge, and m_i the ion mass. Incidentally, the ions are assumed to be singly charged.

B. Fundamental Fields

The fundamental dimensionless fields in our neoclassical, four-field, drift-MHD model are[35,36]

$$\psi(X,\zeta) = \frac{q_s}{\epsilon_s}\,\frac{L_q}{w}\,\frac{A_\parallel}{B_0\,w}, \tag{5}$$

$$N(X,\zeta) = \frac{L_n}{w}\,\frac{\delta n}{n_0}, \tag{6}$$

$$\phi(X,\zeta) = -\frac{\varPhi}{w\,B_0\,V_{*\,i}} + v_p\,X, \tag{7}$$

$$V(X,\zeta) = \frac{\epsilon_s}{q_s}\,\frac{V_{\parallel\,i}}{V_{*\,i}} + v_p, \tag{8}$$

where

$$L_q = 1\left/\left(\frac{d\ln q}{dr}\right)_{r=r_s}\right., \tag{9}$$

$$v_p = \frac{1}{k_\theta\,V_{*\,i}}\,\frac{d\phi_p}{dt}. \tag{10}$$

Here, $A_\parallel$ is the component of the magnetic vector potential parallel to the equilibrium magnetic field (at the rational surface), $L_q > 0$ the safety-factor scale-length at the rational surface, $\varPhi$ the electric scalar potential, v_p the normalized island phase-velocity (which is assumed to be constant in time), and $V_{\parallel\,i}$ the component of the ion fluid velocity parallel to the equilibrium magnetic field (at the rational surface). The four fundamental fields are the normalized helical magnetic flux, the normalized perturbed electron number density, the normalized electric scalar potential, and the normalized parallel ion velocity, respectively. The four fundamental fields are evaluated in a frame of reference that moves with velocity $-(q_s/\epsilon_s)\,v_p\,V_{*\,i}\,\mathbf{e}_\varphi = k_\varphi^{-1}\,(d\phi_p/dt)\,\mathbf{e}_\varphi$ with respect to the laboratory frame, where $\mathbf{e}_\varphi$ is a unit vector pointing in the φ-direction, and $k_\varphi = -n_\varphi/R_0$ the toroidal wavenumber.

C. Neoclassical Four-Field Drift-MHD Model

In the inner region, our neoclassical, four-field, drift-MHD model takes the form[34–36,40]

$$0 = [\phi + \tau\,N, \psi] + \beta\,\eta\,J$$

$$+ \alpha_n^{-1}\,\hat{\nu}_{\theta\,e}\left[\alpha_n^{-1}\,J + V - \partial_X(\phi + \tau\,v_{\theta\,e}\,N) - v_{\theta\,i} - \tau\,v_{\theta\,e}\right], \tag{11}$$

$$0 = [\phi, N] - \rho\,[\alpha_n\,V + J, \psi] - \alpha_c\,\rho\,[\phi + \tau\,N, X] + D\,\partial_X^2 N, \tag{12}$$

$$0 = [\phi, V] - \alpha_n\,(1+\tau)\,[N, \psi] + \mu\,\partial_X^2 V - \hat{\nu}_{\theta\,i}[V - \partial_X(\phi - v_{\theta\,i}\,N)]\,, \tag{13}$$

$$0 = \epsilon\,\partial_X[\phi - N, \partial_X\phi] + [J, \psi] + \alpha_c\,(1+\tau)\,[N, X] + \epsilon\,\mu\,\partial_X^4(\phi - N)$$

$$+ \hat{\nu}_{\theta\,i}\,\partial_X[V - \partial_X(\phi - v_{\theta\,i}\,N)] + \hat{\nu}_{\perp\,i}\,\partial_X[-\partial_X(\phi - v\,N)]\,, \tag{14}$$

where

$$J = \beta^{-1}\left(\partial_X^2\psi - 1\right), \tag{15}$$

$$[A, B] \equiv \partial_X A\;\partial_\zeta B - \partial_\zeta A\;\partial_X B. \tag{16}$$

Furthermore, $\partial_X \equiv (\partial/\partial X)_\zeta$ and $\partial_\zeta \equiv (\partial/\partial\zeta)_X$. Here, Eq. (11) is the parallel Ohm's law, Eq. (12) the electron continuity equation, Eq. (13) the parallel ion equation of motion, and Eq. (14) the parallel ion vorticity equation. The auxiliary field $J(X, \zeta)$ is the normalized perturbed parallel current.

The various dimensionless parameters appearing in Eqs. (11)–(15) have the following definitions:

$$\epsilon = \left(\frac{\epsilon_s}{q_s}\right)^2, \tag{17}$$

$$\rho = \left(\frac{\rho_{\theta\,i}}{w}\right)^2, \tag{18}$$

$$\alpha_n = \frac{L_n/L_q}{\rho}, \tag{19}$$

$$\alpha_c = \frac{2\,(L_n/L_c)}{\rho}, \tag{20}$$

$$\beta = \frac{\beta_i}{\epsilon\,\rho\,\alpha_n^2}, \tag{21}$$

and

$$\eta = \frac{\eta_\parallel}{\mu_0\,k_\theta\,V_{*\,i}\,w^2}, \tag{22}$$

$$D = \left[D_\perp + \beta_i\,(1+\tau)\,\frac{\eta_\perp}{\mu_0}\left(1 - \frac{3}{2}\,\frac{\eta_e}{1+\eta_e}\,\frac{\tau}{1+\tau}\right)\right]\frac{1}{k_\theta\,V_{*\,i}\,w^2}, \tag{23}$$

$$\mu = \frac{\mu_{\perp\,i}}{n_0\,m_i\,k_\theta\,V_{*\,i}\,w^2}, \tag{24}$$

and

$$\hat{\nu}_{\theta\,i} = \left(\frac{\epsilon_s}{q_s}\right)^2 \left(\frac{\nu_{\theta\,i}}{k_\theta\,V_{*\,i}}\right), \tag{25}$$

$$\hat{\nu}_{\perp\,i} = \left(\frac{\epsilon_s}{q_s}\right)^2 \left(\frac{\nu_{\perp\,i}}{k_\theta\,V_{*\,i}}\right), \tag{26}$$

$$\hat{\nu}_{\theta\,e} = \left(\frac{m_e}{m_i}\right) \left(\frac{\epsilon_s}{q_s}\right)^2 \left(\frac{\nu_{\theta\,e}}{k_\theta\,V_{*\,i}}\right), \tag{27}$$

and

$$v_{\theta\,i} = 1 + \lambda_{\theta\,i}\left(\frac{\eta_i}{1+\eta_i}\right) = 1 - 1.172\left(\frac{\eta_i}{1+\eta_i}\right), \tag{28}$$

$$v_{\perp\,i} = 1 + \lambda_{\perp\,i}\left(\frac{\eta_i}{1+\eta_i}\right) = 1 - 2.367\left(\frac{\eta_i}{1+\eta_i}\right), \tag{29}$$

$$v_{\theta\,e} = 1 - \lambda_{\theta\,e}\left(\frac{\eta_e}{1+\eta_e}\right) = 1 - 0.717\left(\frac{\eta_e}{1+\eta_e}\right), \tag{30}$$

and, finally,

$$v = v_{\perp\,i} - v_p. \tag{31}$$

Here, m_e is the electron mass, and L_c the mean radius of curvature of magnetic field-lines at the rational surface.[35,40] The mean curvature is assumed to be favorable (i.e., $L_c > 0$).[41] Note that we are neglecting the geodesic curvature of magnetic field-lines, because this effect cannot be dealt with within the context of a single-helicity calculation.

The quantities $\eta_\parallel$ and $\eta_\perp$ are the parallel and perpendicular plasma resistivities, respectively, whereras $D_\perp$ is a phenomenological perpendicular particle diffusivity (due to small-scale plasma turbulence), and $\mu_{\perp\,i}$ a phenomenological perpendicular ion viscosity (likewise, due to small-scale turbulence). All four of these quantities are evaluated at the rational surface, and are assumed to be constant across the inner region.

D. Model Neoclassical Stress Tensors

The divergence of our model neoclassical ion stress tensor [which is used in the derivation of Eqs. (11)–(15)] takes the form[36,40]

$$\nabla\cdot\pi_i = m_i\,n_0\left[\nu_{\theta\,i}\,V_{\theta\,i}^{nc}\,\mathbf{e}_\theta + \nu_{\perp\,i}\,V_{\perp\,i}^{nc}\,\mathbf{e}_\perp\right], \tag{32}$$

where $\mathbf{e}_\perp = \mathbf{e}_\theta - (\epsilon_s/q_s)\,\mathbf{e}_\varphi$. Here, $\mathbf{e}_\theta$ is a unit vector pointing in the θ-direction, whereas $\mathbf{e}_\perp$ is a unit vector directed perpendicular to the equilibrium magnetic field (at the rational surface). Moreover, $\nu_{\theta\,i}$ and $\nu_{\perp\,i}$ are the neoclassical ion poloidal and perpendicular damping rates, respectively. Finally,

$$V_{\theta\,i}^{nc} = \mathbf{e}_\theta \cdot [\mathbf{V}_i - (1 - v_{\theta\,i})\,\mathbf{V}_{*\,i}]\,, \tag{33}$$

$$V_{\perp\,i}^{nc} = \mathbf{e}_\perp \cdot \left[\mathbf{V}_i - (1 - v_{\perp\,i})\,\mathbf{V}_{*\,i} - v_p\left(\mathbf{V}_{*\,i} - \mathbf{V}_{*\,i}^{(0)}\right)\right], \tag{34}$$

Here, $\mathbf{V}_i$ is the ion fluid velocity (in the laboratory frame), $\mathbf{V}_{*\,i} \equiv (\partial_X N)\,V_{*\,i}\,\mathbf{e}_\perp$ the ion diamagnetic velocity, and $\mathbf{V}_{*\,i}^{(0)} \equiv -V_{*\,i}\,\mathbf{e}_\perp$ the unperturbed (by the island chain) ion diamagnetic velocity.

Note that (in the absence of the island chain) the neoclassical ion stress tensor acts to relax the ion poloidal fluid velocity in the vicinity of the rational surface to the neoclassical value

$$V_{\theta\,i} = (v_{\theta\,i} - 1)\,V_{*\,i} = \lambda_{\theta\,i}\left(\frac{\eta_i\,V_{*\,i}}{1+\eta_i}\right) = -1.172\left(\frac{\eta_i\,T_{i\,0}}{e\,B_0\,L_n}\right), \tag{35}$$

and the ion perpendicular fluid velocity to the neoclassical value

$$V_{\perp\,i} = (v_{\perp\,i} - 1)\,V_{*\,i} = \lambda_{\perp\,i}\left(\frac{\eta_i\,V_{*\,i}}{1+\eta_i}\right) = -2.367\left(\frac{\eta_i\,T_{i\,0}}{e\,B_0\,L_n}\right). \tag{36}$$

Inside the island separatrix (where $\mathbf{V}_{*\,i} = \mathbf{0}$, due to the flattening of the pressure profile), the neoclassical ion stress tensor acts to relax the ion poloidal fluid velocity to zero, so that the island chain is convected by a purely toroidal flow.

Neglecting the effect of plasma impurities, and assuming that the ions lie in the banana collisionality regime, standard neoclassical theory yields $\nu_{\theta\,i} \sim \epsilon_s^{1/2}\nu_i/\epsilon$ and $\lambda_{\theta\,i} = -1.172$, where ν_i is the ion collision frequency.[42] Futhermore, assuming that the ion perpendicular flow damping lies in the so-called "$1/\nu$ regime", neoclassical theory gives $\nu_{\perp\,i} \sim \epsilon_s^{3/2}\,n_\varphi^2\,(T_{i\,0}/m_i)\,(w/R_0)^2/(\epsilon\,R_0^2\,\nu_i)$ and $\lambda_{\perp\,i} = -2.367$.[38,39]

The divergence of our model neoclassical electron stress tensor takes the form

$$\nabla\cdot\pi_e = m_e\,n_0\,\nu_{\theta\,e}\,V_{\theta\,e}^{nc}\,\mathbf{e}_\theta, \tag{37}$$

where $\nu_{\theta\,e}$ is the neoclassical electron poloidal flow damping rate, and

$$V_{\theta\,e}^{nc} = \mathbf{e}_\theta \cdot [\mathbf{V}_e - (1 - v_{\theta\,e})\,\mathbf{V}_{*\,e}]\,. \tag{38}$$

Here, $\mathbf{V}_e$ is the electron fluid velocity, and $\mathbf{V}_{*\,e} \equiv -(\partial_X N)\, V_{*\,e}\, \mathbf{e}_\perp$ the electron diamagnetic velocity. Incidentally, $\nabla\cdot\pi_e$ is neglected with respect to $\nabla\cdot\pi_i$ when both appear in the same equation.

Note that (in the absence of the island chain) the neoclassical electron stress tensor acts to relax the electron poloidal fluid velocity in the vicinity of the rational surface to the neoclassical value

$$V_{\theta\,e} = (1 - v_{\theta\,e})\, V_{*\,e} = \lambda_{\theta\,e}\left(\frac{\eta_e\, V_{*\,e}}{1+\eta_e}\right) = 0.717\left(\frac{\eta_e\, T_{e\,0}}{e\, B_0\, L_n}\right). \tag{39}$$

Inside the island separatrix (where $\mathbf{V}_{*\,e} = \mathbf{0}$, due to the flattening of the pressure profile), the neoclassical electron stress tensor acts to relax the electron poloidal fluid velocity to zero,

Assuming that the electrons lie in the banana collisionality regime, standard neoclassical theory yields $\nu_{\theta\,e} \sim \epsilon_s^{1/2}\, \nu_e/\epsilon$ and $\lambda_{\theta\,e} = +0.717$, where ν_e is the electron collision frequency.[43]

Roughly speaking, our expressions for the divergences of the neoclassical ion and electron stress tensors are the flux-surface averages of the true divergences. This approximate treatment of the divergences is necessary within the context of a single-helicity calculation.

E. Boundary Conditions

Equations (11)–(15) are subject to the boundary conditions[35,36]

$$\psi(X,\zeta) \to \frac{1}{2}\, X^{\,2} + \cos\zeta, \tag{40}$$

$$N(X,\zeta) \to -X, \tag{41}$$

$$\phi(X,\zeta) \to -v\, X, \tag{42}$$

$$V(X,\zeta) \to v_{\theta\,i} - v, \tag{43}$$

$$J(X,\zeta) \to 0, \tag{44}$$

as $|X| \to \infty$. It follows that the fields $\psi(X,\zeta)$, $V(X,\zeta)$, and $J(X,\zeta)$ are even in X, whereas the fields $N(X,\zeta)$ and $\phi(X,\zeta)$ are odd. Of course, all fields are periodic in ζ with period 2π.

F. Island Phase-Velocity Parameter

The dimensionless parameter v, defined in Eq. (31), has a simple physical interpretation. If $v = -\tau$ then the island chain co-rotates with the unperturbed electron fluid at the rational surface; if $v = 0$ then the island chain co-rotates with the unperturbed guiding-center fluid; and, if $v = +1$ then the island chain co-rotates with the unperturbed ion fluid.

G. Island Geometry

To lowest order, we expect that[34–36]

$$\psi(X,\zeta) = \Omega(X,\zeta) \equiv \frac{1}{2}\,X^{\,2} + \cos\zeta \tag{45}$$

in the inner region. In fact, this result, which is known as the constant-ψ approximation,[5] holds as long as $\beta \ll 1$. The contours of $\Omega(X,\zeta)$ map out the magnetic flux-surfaces of a helical magnetic island chain whose O-points are located at $X = 0$ and $\zeta = \pi$, and whose X-points are located at $X = 0$ and $\zeta = 0$. The magnetic separatrix corresponds to $\Omega = 1$, the region enclosed by the separatrix to $-1 \leq \Omega < 1$, and the region outside the separatrix to $\Omega > 1$.

H. Flux-Surface Average Operator

The flux-surface average operator, $\langle\cdots\rangle$, is defined as the annihilator of $[A,\Omega]$. In other words, $\langle[A,\Omega]\rangle = 0$, for any field $A(X,\zeta)$. It follows that

$$\langle A(s,\Omega,\zeta)\rangle = \oint \frac{A(s,\Omega,\zeta)}{[2\,(\Omega-\cos\zeta)]^{\,1/2}}\,\frac{d\zeta}{2\pi} \tag{46}$$

for $1 \leq \Omega$, and

$$\langle A(s,\Omega,\zeta)\rangle = \int_{\zeta_0}^{2\pi-\zeta_0} \frac{A(s,\Omega,\zeta)+A(-s,\Omega,\zeta)}{2\,[2\,(\Omega-\cos\zeta)]^{\,1/2}}\,\frac{d\zeta}{2\pi} \tag{47}$$

for $-1 \leq \Omega < 1$. Here, $s = \mathrm{sgn}(X)$ and $\zeta_0 = \cos^{-1}(\Omega)$, where $0 \leq \zeta_0 \leq \pi$.

It is helpful to define

$$\tilde{A} \equiv A - \frac{\langle A\rangle}{\langle 1\rangle}. \tag{48}$$

It follows that $\langle \tilde{A} \rangle = 0$, for any field $A(X, \zeta)$. It is also easily demonstrated that $\langle [A, F(\Omega)] \rangle = 0$, for any function $F(\Omega)$.

I. Asymptotic Matching

Standard asymptotic matching between the inner and outer regions[6,44,45] yields the island width evolution equation,

$$4\, I_1\, \tau_R\, \frac{d}{dt}\left(\frac{w}{r_s}\right) = \Delta'\, r_s + 2\, m_\theta \left(\frac{w_v}{w}\right)^2 \cos\phi_p + J_c\, \beta\, \frac{r_s}{w}, \tag{49}$$

and the island phase evolution equation,

$$\frac{d^2\phi_p}{dt^2} \propto -2\, m_\theta \left(\frac{w_v}{w}\right)^2 \sin\phi_p + J_s\, \beta\, \frac{r_s}{w}. \tag{50}$$

Here, $I_1 = 0.823$ (see Appendix), $\tau_R = \mu_0\, r_s^2/\eta_\parallel$, and

$$J_c = -2 \int_{-\infty}^{\infty} J\, \cos\zeta\, dX\, \frac{d\zeta}{2\pi} = -4 \int_{-1}^{\infty} \langle J\, \cos\zeta \rangle\, d\Omega, \tag{51}$$

$$J_s = -2 \int_{-\infty}^{\infty} J\, \sin\zeta\, dX\, \frac{d\zeta}{2\pi} = -4 \int_{-1}^{\infty} \langle X\, [J, \Omega] \rangle\, d\Omega. \tag{52}$$

Note that, for the sake of completeness, we have taken into account the possibility that the plasma is subject to a static, external, magnetic perturbation possessing the same helicity as the island chain. Here, $4\, w_v$ is the full radial width of the vacuum island chain (i.e., the island chain obtained by naively superimposing the vacuum magnetic perturbation onto the unperturbed plasma equilibrium), and ϕ_p becomes the helical phase-shift between the true island chain and the vacuum island chain.

The first term on the right-hand side of Eq. (49) governs the intrinsic stability of the island chain. (The chain is intrinsically stable if $\Delta' < 0$, and vice versa.) The second term represents the destabilizing effect of the external perturbation. The final term represents the destabilizing or stabilizing (depending on whether the integral J_c is positive or negative, respectively) effect of helical currents flowing in the inner region.

The first term on the right-hand side of Eq. (50) represents the electromagnetic locking torque exerted on the plasma in the inner region by the external perturbation. The second term represents the drag torque due to the combined effects of neoclassical ion poloidal flow damping, neoclassical ion toroidal flow damping, and perpendicular ion viscosity.

J. Expansion Procedure

Equations (11)–(15) are solved, subject to the boundary conditions (40)–(44), via an expansion in two small parameters, Δ and δ, where $\Delta \lll \delta \ll 1$. The expansion procedure is as follows. First, the coordinates X and ζ are assumed to be $\mathcal{O}(\Delta^0\,\delta^0)$. Next, some particular ordering scheme is adopted for the fifteen physics parameters $v_{\theta\,i}$, $v_{\theta\,e}$, v, τ, α_n, α_c, ϵ, ρ, β, $\hat{\nu}_{\theta\,i}$, $\hat{\nu}_{\perp\,i}$, $\hat{\nu}_{\theta\,e}$, η, D, and μ. The fields ψ, N, ϕ, V, and J are then expanded in the form $\psi(X,\zeta) = \sum_{i,j=0,\infty}\psi_{i,j}(X,\zeta)$, et cetera, where $\psi_{i,j}\sim\mathcal{O}(\Delta^i\,\delta^j)$. Finally, Eqs. (11)–(15) are solved order by order, subject to the boundary conditions (40)–(44).

III. WEAK NEOCLASSICAL ION POLOIDAL FLOW DAMPING REGIME

A. Ordering Scheme

The ordering scheme adopted in the so-called weak neoclassical ion poloidal flow damping regime is: [35,36]

$\Delta^0\,\delta^0$: $v_{\theta\,i}$, $v_{\theta\,e}$, v, τ, α_n,

$\Delta^0\,\delta^1$: α_c, ϵ, ρ, β,

$\Delta^1\,\delta^0$: $\hat{\nu}_{\theta\,i}$, $\hat{\nu}_{\perp\,i}$, η, D, μ,

$\Delta^1\,\delta^2$: $\hat{\nu}_{\theta\,e}$.

This ordering scheme is suitable for a constant-ψ (i.e., $\beta \ll 1$) magnetic island chain whose radial width is much larger than the ion poloidal gyroradius (i.e., $\rho \ll 1$), and which is embedded in a large aspect-ratio (i.e., $\epsilon \ll 1$), high-temperature (i.e., η, D, $\mu \lll 1$) tokamak plasma equilibrium. The defining feature of the weak neoclassical ion poloidal flow damping regime is that the ion poloidal flow damping rate is sufficiently small that the neoclassical ion stress tensor is not the dominant term in the ion parallel equation of motion.

B. Order $\Delta^0\,\delta^0$

To order $\Delta^0\,\delta^0$, Eqs. (11)–(15) yield

$$0 = [\phi_{0,0} + \tau\,N_{0,0}, \psi_{0,0}], \tag{53}$$

$$0 = [\phi_{0,0}, N_{0,0}], \tag{54}$$

$$0 = [\phi_{0,0}.V_{0,0}] - \alpha_n\,(1+\tau)\,[N_{0,0}, \psi_{0,0}], \tag{55}$$

$$0 = [J_{0,0}, \psi_{0,0}], \tag{56}$$

$$\partial_X^2\psi_{0,0} = 1. \tag{57}$$

Equations (40), (45), and (57) give

$$\psi_{0,0} = \Omega(X, \zeta). \tag{58}$$

Equations (41), (42), (53), and (54) imply that

$$\phi_{0,0} = s\,\phi_0(\Omega), \tag{59}$$

$$N_{0,0} = s\,N_0(\Omega). \tag{60}$$

Note that, by symmetry, $\phi_0 = N_0 = 0$ inside the separatrix, which means that the electron number density and temperature profiles are flattened inside the separatrix. Let

$$M(\Omega) = -\frac{d\phi_0}{d\Omega}, \tag{61}$$

$$L(\Omega) = -\frac{dN_0}{d\Omega}. \tag{62}$$

Equations (41) and (42) yield

$$M(\Omega\to\infty) = \frac{v}{\sqrt{2\,\Omega}}, \tag{63}$$

$$L(\Omega\to\infty) = \frac{1}{\sqrt{2\,\Omega}}. \tag{64}$$

Again, by symmetry, $M = L = 0$ inside the separatrix. Equations (55), (61), and (62) give

$$V_{0,0} = V_0(\Omega), \tag{65}$$

and Eq. (43) implies that

$$V_0(\Omega \to \infty) = v_{\theta\,i} - v. \tag{66}$$

Finally, Eq. (56) yields

$$J_{0,0} = 0. \tag{67}$$

C. Order $\Delta^0\,\delta^{\,1}$

To order $\Delta^0\,\delta^{\,1}$, Eqs. (14), (59), (60), and (67) give

$$[J_{1,0}, \Omega] = -\epsilon\,\partial_X[\phi_0 - N_0, \partial_X\phi_0] - \alpha_c\,(1+\tau)\,[N_0, |X|]. \tag{68}$$

It follows, with the aid of Eqs. (61) and (62), that

$$[J_{0,1}, \Omega] = \left[\frac{\epsilon}{2}\,d_\Omega\{(M-L)\,M\}X^{\,2} - \alpha_c\,(1+\tau)\,L\,|X|, \Omega\right], \tag{69}$$

where $d_\Omega \equiv d/d\Omega$. Hence,

$$J_{0,1} = \frac{\epsilon}{2}\,d_\Omega[(M-L)\,M]\,\widetilde{X^{\,2}} - \alpha_c\,(1+\tau)\,L\,\widetilde{|X|} + \bar{J}(\Omega), \tag{70}$$

where $\bar{J}(\Omega)$ is an arbitrary flux function. However, the lowest-order flux-surface average of Eq. (11) implies that

$$\bar{J}(\Omega) = -\alpha_n\,\epsilon\,\nu_{\theta\,e}\,\tau_e\left(V_0 + \frac{M + \tau\,v_{\theta\,e}\,L}{\langle 1\rangle} - v_{\theta\,i} - \tau\,v_{\theta\,e}\right), \tag{71}$$

where

$$\tau_e = \nu_e^{-1} = \frac{m_e}{n_0\,e^{\,2}\,\eta_\parallel} \tag{72}$$

is the electron collision time.

Finally, it is easily demonstrated that

$$X\,[J_{0,1}, \Omega] = \frac{\epsilon}{6}\,[X^{\,3}, (M-L)\,M] + \frac{1}{2}\,\alpha_c\,(1+\tau)\,[s\,X^{\,2}, N_0], \tag{73}$$

which implies that

$$\langle X\,[J_{0,1}, \Omega]\rangle = 0. \tag{74}$$

In other words, $J_{0,1}$ does not contribute to the torque integral, J_s [see Eq. (52)]. Thus, in order to calculate J_s, and, hence, to determine the phase-velocity of a freely rotating island chain, we must expand to higher order.

D. Order $\Delta^1\,\delta^0$

To order $\Delta^1\,\delta^0$, Eqs. (11)–(15), (58)–(60), (65), and (67) yield

$$0 = [\phi_{1,0}+\tau\,N_{1,0},\mathit{\Omega}] + s\,[\phi_0+\tau\,N_0,\psi_{1,0}], \tag{75}$$

$$0 = s\,[\phi_{1,0},N_0] + s\,[\phi_0,N_{1,0}] + s\,D\,\partial_X^2 N_0, \tag{76}$$

$$\begin{aligned}0 &= [\phi_{1,0},V_0] + s\,[\phi_0,V_{1,0}] - \alpha_n\,(1+\tau)\,[N_{1,0},\mathit{\Omega}] - s\,\alpha_n\,(1+\tau)\,[N_0,\psi_{1,0}]\\ &\quad + \mu\,\partial_X^2 V_0 - \hat{\nu}_{\theta\,i}\,[V_0 - s\,\partial_X(\phi_0 - v_{\theta\,i}\,N_0)]\,,\end{aligned} \tag{77}$$

$$0 = [J_{1,0},\mathit{\Omega}] + \hat{\nu}_{\theta\,i}\,\partial_X[V_0 - s\,\partial_X(\phi_0 - v_{\theta\,i}\,N_0)] + \hat{\nu}_{\perp\,i}\,\partial_X[-s\,\partial_X(\phi_0 - v\,N_0)]\,, \tag{78}$$

$$\partial_X^2\psi_{1,0} = 0. \tag{79}$$

It follows from Eq. (79) that

$$\psi_{1,0} = 0, \tag{80}$$

from Eq. (75) that

$$\phi_{1,0} = -\tau\,N_{1,0}, \tag{81}$$

from Eqs. (61), (62), and (76) that

$$[N_{1,0},\mathit{\Omega}] = D\left(\frac{X^{\,2}\,d_{\mathit{\Omega}}L + L}{M+\tau\,L}\right), \tag{82}$$

from Eq. (77) that

$$\begin{aligned}[\{\tau\,d_{\mathit{\Omega}}V_0 + \alpha_n\,(1+\tau)\}N_{1,0} - s\,M\,V_{1,0},\mathit{\Omega}] &= \mu\,X\,\partial_{\mathit{\Omega}}(X\,d_{\mathit{\Omega}}V_0)\\ &\quad - \hat{\nu}_{\theta\,i}\,[V_0 + |X|\,(M - v_{\theta\,i}\,L)]\,,\end{aligned} \tag{83}$$

and from Eq. (78) that

$$[J_{1,0},\mathit{\Omega}] = -\hat{\nu}_{\theta\,i}\,\partial_X[V_0 + |X|\,(M - v_{\theta\,i}\,L)] - \hat{\nu}_{\perp\,i}\,\partial_X[|X|\,(M - v\,L)]\,. \tag{84}$$

Here, $\partial_{\mathit{\Omega}} \equiv (\partial/\partial\mathit{\Omega})_\zeta$.

Given that $M = L = 0$ within the island separatrix, the previous four equations suggest that $\phi_{1,0} = N_{1,0} = V_{1,0} = J_{1,0} = V_0 = 0$ in this region. In particular, the flux-surface average of Eq. (84) implies that $d_\Omega V_0 = 0$ within the separatrix. The flux-surface average of Eq. (83) then reveals that $V_0 = 0$ in this region.

The flux-surface average of Eq. (82) yields

$$L(\Omega) = \begin{cases} 1/\langle X^2\rangle & 1 \leq \Omega \\ 0 & -1 \leq \Omega < 1 \end{cases} \tag{85}$$

The flux-surface average of Eq. (84) gives

$$\hat{\nu}_{\theta\,i}\, d_\Omega V_0 = -d_\Omega\big[\hat{\nu}_{\theta\,i}\,\langle X^2\rangle\,(M - v_{\theta\,i}\,L) + \hat{\nu}_{\perp\,i}\,\langle X^2\rangle\,(M - v\,L)\big] \tag{86}$$

outside the magnetic separatrix. It follows from Eqs. (63), (64), (66), and (85) that

$$V_0(\Omega) = -\left(\frac{\hat{\nu}_{\theta\,i} + \hat{\nu}_{\perp\,i}}{\hat{\nu}_{\theta\,i}}\right)\big(\langle X^2\rangle\,F + \bar{v}\big) \tag{87}$$

outside the separatrix, where

$$\bar{v} = \frac{\hat{\nu}_{\theta\,i}\,(1 - v_{\theta\,i}) + \hat{\nu}_{\perp\,i}\,(1 - v)}{\hat{\nu}_{\theta\,i} + \hat{\nu}_{\perp\,i}}, \tag{88}$$

and

$$F(\Omega) \equiv M(\Omega) - L(\Omega). \tag{89}$$

Note that $F = 0$ inside the magnetic separatrix. The viscous term in Eq. (83) requires continuity of $V_0(\Omega)$ across the separatrix. Given that $V_0 = 0$ inside the separatrix, and $\langle X^2\rangle = 4/\pi$ on the separatrix (see Appendix), Eq. (87) yields

$$F(1) = -\frac{\pi}{4}\,\bar{v}. \tag{90}$$

Finally, Eqs. (63), (64), and (89) give

$$F(\Omega \to \infty) = \frac{v - 1}{\sqrt{2\,\Omega}}. \tag{91}$$

The flux-surface average of Eq. (83) yields

$$0 = \mu\, d_\Omega\big(\langle X^2\rangle\, d_\Omega V_0\big) - \hat{\nu}_{\theta\,i}\, V_0\,\langle 1\rangle - \hat{\nu}_{\theta\,i}\left(F + \frac{1 - v_{\theta\,i}}{\langle X^2\rangle}\right) \tag{92}$$

outside the magnetic separatrix. It follows from Eq. (87) that [35,36]

$$0 = \hat{\mu}\, d_\Omega\big[\langle X^{\,2}\rangle\, d_\Omega(\langle X^{\,2}\rangle F)\big] - \hat{\nu}_{\theta\, i}\left(\langle X^{\,2}\rangle\langle 1\rangle - 1\right)\left(F + \frac{1-v_{\theta\, i}}{\langle X^{\,2}\rangle}\right)$$

$$- \hat{\nu}_{\perp\, i}\left(\langle X^{\,2}\rangle F + 1 - v\right)\langle 1\rangle, \tag{93}$$

where

$$\hat{\mu} = \left(\frac{\hat{\nu}_{\theta\, i} + \hat{\nu}_{\perp\, i}}{\hat{\nu}_{\theta\, i}}\right)\mu. \tag{94}$$

E. Evaluation of J_c

According to Eqs. (70), (71), (85), and (87)–(89),

$$J_{0,1} = \frac{\epsilon}{2}\, d_\Omega\left[F\left(F + \frac{1}{\langle X^{\,2}\rangle}\right)\right]\widetilde{X^{\,2}} - \alpha_c\,(1+\tau)\,\frac{\widetilde{|X|}}{\langle X^{\,2}\rangle}$$

$$+ \alpha_n\,\epsilon\,\nu_{\theta\, e}\,\tau_e\left[\left(\langle X^{\,2}\rangle - \frac{1}{\langle 1\rangle}\right)F + \frac{\hat{\nu}_{\perp\, i}}{\hat{\nu}_{\theta\, i}}\left(\langle X^{\,2}\rangle F + 1 - v\right)\right.$$

$$\left. + (1 + \tau\, v_{\theta\, e})\left(1 - \frac{1}{\langle 1\rangle\langle X^{\,2}\rangle}\right)\right] \tag{95}$$

for $\Omega \geq 1$, and

$$J_{0,1} = \alpha_n\,\epsilon\,\nu_{\theta\, e}\,\tau_e\,(v_{\theta\, i} + \tau\, v_{\theta\, e}) \tag{96}$$

for $-1 \leq \Omega < 1$. Thus, it follows from Eqs. (45) and (51) that

$$J_c = J_p + J_g + J_b, \tag{97}$$

where

$$J_p = \epsilon \int_{1-}^{\infty} d_\Omega\left[F\left(F + \frac{1}{\langle X^{\,2}\rangle}\right)\right]\langle\widetilde{X^{\,2}}\,\widetilde{X^{\,2}}\rangle\, d\Omega \tag{98}$$

parameterizes the effect of the perturbed ion polarization current on island stability, whereas

$$J_g = -\alpha_c\,(1+\tau)\int_1^{\infty} 2\,\frac{\langle\widetilde{|X|}\,\widetilde{X^{\,2}}\rangle}{\langle X^{\,2}\rangle}\, d\Omega \tag{99}$$

parameterizes the effect of magnetic field-line curvature on island stability, and, finally,

$$J_b = -\alpha_n\,\epsilon\,\nu_{\theta\, e}\,\tau_e\int_1^{\infty} 2\left\{\left(\langle X^{\,2}\rangle - \frac{1}{\langle 1\rangle}\right)F + \frac{\hat{\nu}_{\perp\, i}}{\hat{\nu}_{\theta\, i}}\left(\langle X^{\,2}\rangle F + 1 - v\right)\right.$$

$$\left. + (1+\tau\, v_{\theta\, e})\left(1 - \frac{1}{\langle 1\rangle\langle X^{\,2}\rangle}\right) - v_{\theta\, i} - \tau\, v_{\theta\, e}\right\}\left(2\,\Omega\,\langle 1\rangle - \langle X^{\,2}\rangle\right) d\Omega \tag{100}$$

parameterizes the effect of the perturbed bootstrap current on island stability. In deriving the previous expressions, we have made use of the following easily demonstrated results: $\langle\widetilde{A}\,\cos\zeta\rangle = -(1/2)\,\langle\widetilde{A}\,\widetilde{X^{\,2}}\rangle$ and $\int_{-1}^{\infty}\langle\cos\zeta\rangle\,d\Omega = 0$.

F. Evaluation of J_s

Equations (84), (85), and (87)–(89) imply that

$$[J_{1,0},\Omega] = -\partial_X G, \tag{101}$$

where

$$G = -\hat{\nu}_{\theta\,i}\left(\langle X^{\,2}\rangle - |X|\right)\left(F + \frac{1-v_{\theta\,i}}{\langle X^{\,2}\rangle}\right) - \hat{\nu}_{\perp\,i}\left(\langle X^{\,2}\rangle - |X|\right)\left(F + \frac{1-v}{\langle X^{\,2}\rangle}\right) \tag{102}$$

for $\Omega \geq 1$, and $G = 0$ for $-1 \leq \Omega < 1$. Note that G is continuous across the separatrix ($\Omega = 1$). It follows that [35,36]

$$\langle X\,[J_{1,0},\Omega]\rangle = -d_\Omega\langle X^{\,2}\,G\rangle + \langle G\rangle. \tag{103}$$

Hence, Eq. (52) yields

$$J_s = -4\int_1^\infty \langle G\rangle\,d\Omega, \tag{104}$$

because $\langle X^{\,2}\,G\rangle_{\Omega\to\infty} = 0$. Thus, we obtain

$$J_s = \hat{\nu}_{\theta\,i}\int_1^\infty 4\left(\langle 1\rangle\langle X^{\,2}\rangle - 1\right)\left(F + \frac{1-v_{\theta\,i}}{\langle X^{\,2}\rangle}\right)d\Omega + \hat{\nu}_{\perp\,i}\int_1^\infty 4\left(\langle 1\rangle\langle X^{\,2}\rangle - 1\right)\left(F + \frac{1-v}{\langle X^{\,2}\rangle}\right)d\Omega. \tag{105}$$

G. Transformed Equations

Let

$$Y(k) = -2\,k\left[F(k) + \frac{1-v_{\theta\,i}}{2\,k\,\mathcal{C}(k)}\right]\Big/\,(v_{\theta\,i} - v), \tag{106}$$

where $k = [(1+\Omega)/2]^{\,1/2}$. Note that $k=0$ corresponds to the island O-point, $k=1$ to the island separatrix, and $k\to\infty$ to $\Omega\to\infty$. Here, $\mathcal{C}(k)$ is defined in the Appendix. It follows from Eqs. (88), (90), and (91) that

$$Y(1) = \frac{\pi}{2}\left(\frac{\hat{\nu}_{\perp\,i}}{\hat{\nu}_{\theta\,i}+\hat{\nu}_{\perp\,i}}\right), \tag{107}$$

$$Y(k\rightarrow\infty) = 1. \tag{108}$$

Furthermore, Eq. (93) reduces to

$$0 = \frac{\hat{\mu}}{4}\,d_k[\mathcal{C}\,d_k(\mathcal{C}\,Y)] - \hat{\nu}_{\theta\,i}\,(\mathcal{A}\,\mathcal{C}-1)\,Y - \hat{\nu}_{\perp\,i}\,(\mathcal{C}\,Y-1)\,\mathcal{A}, \tag{109}$$

where $d_k \equiv d/dk$, and $\mathcal{A}(k)$ is defined in the Appendix. Equations (98)–(100) yield

$$J_p = \epsilon\int_{1-}^{\infty}\frac{d}{dk}\left[\left\{(v_{\theta\,i}-v)\,\frac{Y}{2\,k}+\frac{1-v_{\theta\,i}}{2\,k\,\mathcal{C}}\right\}\left\{(v_{\theta\,i}-v)\,\frac{Y}{2\,k}-\frac{v_{\theta\,i}}{2\,k\,\mathcal{C}}\right\}\right]8\left(\mathcal{E}-\frac{\mathcal{C}^{\,2}}{\mathcal{A}}\right)k^{\,3}\,dk, \tag{110}$$

$$J_g = -\alpha_c\,(1+\tau)\int_1^{\infty}16\left(\frac{\mathcal{D}}{B}-\frac{1}{\mathcal{A}}\right)k^{\,2}\,dk, \tag{111}$$

$$J_b = \alpha_n\,\epsilon\,\nu_{\theta\,e}\,\tau_e\,[1-v-\tau\,(1-v_{\theta\,e})]\int_1^{\infty}16\left[\left\{1+\frac{\hat{\nu}_{\perp\,i}}{\hat{\nu}_{\theta\,i}}-\frac{1}{\mathcal{A}\,\mathcal{C}}\right\}\mathcal{C}\,Y-\frac{\hat{\nu}_{\perp\,i}}{\hat{\nu}_{\theta\,i}}\right](\mathcal{D}\,\mathcal{A}-\mathcal{C})\,k^{\,2}\,dk$$
$$+\,\alpha_n\,\epsilon\,\nu_{\theta\,e}\,\tau_e\,(v_{\theta\,i}+\tau\,v_{\theta\,e})\int_1^{\infty}16\left(\frac{\mathcal{D}}{\mathcal{C}}-\frac{1}{\mathcal{A}}\right)k^{\,2}\,dk, \tag{112}$$

where $\mathcal{D}(k)$ and $\mathcal{E}(k)$ are defined in the Appendix. Finally, Eq. (105) gives

$$J_s = -\hat{\nu}_{\theta\,i}\,(v_{\theta\,i}-v)\int_1^{\infty}8\,(\mathcal{A}\,\mathcal{C}-1)\,Y\,dk - \hat{\nu}_{\perp\,i}\,(v_{\theta\,i}-v)\int_1^{\infty}8\,(\mathcal{A}\,\mathcal{C}-1)\left(Y-\frac{1}{\mathcal{C}}\right)dk. \tag{113}$$

It remains to solve Eq. (109), subject to the boundary conditions (107) and (108), and then to evaluate the integrals (110)–(113). This task, which involves the elimination of an unphysical solution that varies as $Y \sim \exp[+2\,(\hat{\nu}_{\perp\,i}/\hat{\mu})^{1/2}\,k]$ as $k\rightarrow\infty$, can be achieved analytically in five different parameter regimes that are described in Sect. III I.[36]

H. Separatrix Boundary Layer

The flux-surface functions $M(\mathit{\Omega})$ and $L(\mathit{\Omega})$ are both zero inside, and non-zero just outside, the magnetic separatrix. Retaining selected higher-order terms (containing radial derivatives) in Eqs. (76) and (78), we find that

$$(M+\tau\,L)\,[N_{1,0},\mathit{\Omega}]-s\,\rho\,[J_{1,0},\mathit{\Omega}]\simeq D\left(X^{\,2}\,d_{\mathit{\Omega}}L+L\right), \tag{114}$$

$$s\,[J_{1,0},\mathit{\Omega}]\simeq-\epsilon\,\partial_X[\phi_0-N_0,\partial_X\phi_{1,0}]=\epsilon\,\tau\,(M-L)\,\partial_X^2[N_{1,0},\mathit{\Omega}], \tag{115}$$

so that Eq. (82) generalizes to give

$$\left\{M+\tau\,L-\tau\,(M-L)\,\epsilon\,\rho\,\partial_X^2\right\}[N_{1,0},\mathit{\Omega}]\simeq D\left(X^{\,2}\,d_{\mathit{\Omega}}L+L\right), \tag{116}$$

which suggests that the apparent discontinuities in the functions $M(\mathit{\Omega})$ and $L(\mathit{\Omega})$ are resolved in a thin boundary layer, centered on the magnetic separatrix, of (unnormalized) width $(\epsilon\,\rho)^{1/2}\,w=\rho_i$, where $\rho_i=(\epsilon_s/q_s)\,\rho_{\theta\,i}$ is the ion gyroradius.[47]

Equation (110) can be written

$$J_p=\epsilon\int_{1-}^{\infty}d_k\,[F\,(F+L)]\,8\left(\mathcal{E}-\frac{\mathcal{C}^{\,2}}{\mathcal{A}}\right)k^{\,3}\,dk. \tag{117}$$

In accordance with Eqs. (85) and (90), let us suppose that, in the immediate vicinity of the separatrix, $L(k)$ and $F(k)\equiv M(k)-L(k)$ take the following forms:

$$L(k)=\frac{f(k)}{2\,k\,\mathcal{C}(k)}, \tag{118}$$

$$F(k)=-\frac{\bar{v}\,f(k)}{2\,k\,\mathcal{C}(k)}, \tag{119}$$

where

$$f(k)=\frac{1}{2}\left(1+\tanh\left[(k-1)\,\frac{2\,w}{\rho_i}\right]\right). \tag{120}$$

In effect, we have resolved the discontinuities in the functions $L(k)$ and $F(k)$ across the separatrix in a boundary layer of (unnormalized) thickness ρ_i. In the limit that $\rho_i/w\ll 1$, the contribution of the boundary layer to the polarization integral (117) can be written

$$J_{p\,s}=\left[\frac{2\pi}{3}-Q\left(\frac{\rho_i}{w}\right)\right]\epsilon\,\bar{v}\,(\bar{v}-1), \tag{121}$$

where

$$Q(x)=2\pi\int_0^{\infty}\frac{\mathrm{sech}^2(y)}{\ln(16/x)+\ln(1/y)}\,dy\simeq\frac{6.2}{\ln(16/x)}-\frac{3.0}{\ln^{\,2}(16/x)}. \tag{122}$$

In deriving the previous equation, we have made use of the fact that the functions $\mathcal{C}(k)$ and $\mathcal{E}(k)$ are well behaved as $k\rightarrow 1$, whereas the function $\mathcal{A}(k)$ has a logarithmic singularity.[48] The separatrix boundary layer response function, $Q(x)$, is plotted in Fig. 1.

According to the previous analysis, the polarization integral, (110), takes the form

$$J_p = \left[\frac{2\pi}{3} - Q\left(\frac{\rho_i}{w}\right)\right]\epsilon\,\bar{v}\,(\bar{v}-1) \tag{123}$$
$$+\,\epsilon\int_{1+}^{\infty}\frac{d}{dk}\left[\left\{(v_{\theta\,i}-v)\,\frac{Y}{2\,k}+\frac{1-v_{\theta\,i}}{2\,k\,\mathcal{C}}\right\}\left\{(v_{\theta\,i}-v)\,\frac{Y}{2\,k}-\frac{v_{\theta\,i}}{2\,k\,\mathcal{C}}\right\}\right]8\left(\mathcal{E}-\frac{\mathcal{C}^{\,2}}{\mathcal{A}}\right)k^{\,3}\,dk.$$

Of course, the first term on the right-hand side of the previous equation emanates from the separatrix boundary layer.[25] Note that the neglect of the finite thickness of the boundary layer leads to a significant overestimate of the contribution of the layer to the polarization integral.[26,27]

I. Island Solution Regimes

The extents of the five analytic solution regimes, mentioned in Sect. III G, in the $\hat{\nu}_{\perp\,i}$–$\hat{\nu}_{\theta\,i}$ plane are indicated in Fig. 2.

Regime I corresponds to $\hat{\nu}_{\perp\,i}\gg\hat{\nu}_{\theta\,i}$ and $\mu\ll\hat{\nu}_{\theta\,i}$. In this regime,

$$Y(k)\simeq\frac{1}{\mathcal{C}}\left[1-\frac{\hat{\nu}_{\theta\,i}}{\hat{\nu}_{\perp\,i}}\left(1-\frac{1}{\mathcal{A}\,\mathcal{C}}\right)\right]. \tag{124}$$

It follows that

$$J_p = -\left[I_2 - Q\left(\frac{\rho_i}{w}\right)\right]\epsilon\,v\,(1-v), \tag{125}$$

$$J_g = -I_3\,\alpha_c\,(1+\tau), \tag{126}$$

$$J_b = I_3\,\alpha_n\,\epsilon\,\nu_{\theta\,e}\,\tau_e\,(v_{\theta\,i}+\tau\,v_{\theta\,e}), \tag{127}$$

$$J_s = -I_4\,\hat{\nu}_{\theta\,i}\,(v_{\theta\,i}-v), \tag{128}$$

where $I_2=1.38$, $I_3=1.58$, and $I_4=0.357$ are defined in the Appendix.

Regime II corresponds to $\hat{\nu}_{\theta\,i}\gg\hat{\nu}_{\perp\,i}$ and $\mu\ll(\hat{\nu}_{\theta\,i}\,\hat{\nu}_{\perp\,i})^{1/2}$. In this regime,

$$Y(k)\simeq\frac{\hat{\nu}_{\perp\,i}}{\hat{\nu}_{\theta\,i}}\,\frac{\mathcal{A}}{\mathcal{A}\,\mathcal{C}\,(1+\hat{\nu}_{\perp\,i}/\hat{\nu}_{\theta\,i})-1}. \tag{129}$$

It follows that

$$J_p = -\left[I_2 - Q\left(\frac{\rho_i}{w}\right)\right]\epsilon\, v_{\theta\,i}\,(1-v_{\theta\,i}), \tag{130}$$

$$J_g = -I_3\,\alpha_c\,(1+\tau), \tag{131}$$

$$J_b = I_3\,\alpha_n\,\epsilon\,\nu_{\theta\,e}\,\tau_e\,(v_{\theta\,i}+\tau\,v_{\theta\,e}), \tag{132}$$

$$J_s = -I_5\,\hat{\nu}_{\theta\,i}^{\,1/4}\,\hat{\nu}_{\perp\,i}^{\,3/4}\,(v_{\theta\,i}-v), \tag{133}$$

where $I_5 = 3.74$ is defined in the Appendix.

Regime IIIa corresponds to $\hat{\nu}_{\theta\,i} \gg \mu$ and $\mu \gg (\hat{\nu}_{\theta\,i}\,\hat{\nu}_{\perp\,i})^{1/2}$. In this regime,

$$Y(k) \simeq \frac{\hat{\nu}_{\perp\,i}}{\hat{\nu}_{\theta\,i}}\,\frac{\mathcal{A}}{\mathcal{A}\,\mathcal{C}\,(1+\hat{\nu}_{\perp\,i}/\hat{\nu}_{\theta\,i})-1} \tag{134}$$

for $1 < k \ll k_1$, and

$$Y(k) \simeq 1 - \left(1+\frac{k_1}{k_2}\right){\rm e}^{-k/k_2} \tag{135}$$

for $k \gtrsim k_1$. Here, $k_1 = (\hat{\nu}_{\theta\,i}/8\,\mu)^{1/2}$ and $k_2 = (\mu/4\,\hat{\nu}_{\perp\,i})^{1/2}$. Regime IIIb corresponds to $\hat{\nu}_{\theta\,i} \gg \hat{\nu}_{\perp\,i}$ and $\mu \gg \hat{\nu}_{\theta\,i}$. In this regime,

$$Y(k) \simeq \frac{\hat{\nu}_{\perp\,i}}{\hat{\nu}_{\theta\,i}\,\mathcal{C}} \tag{136}$$

for $1 < k \ll k_3$, and

$$Y(k) \simeq 1 - \left(1+\frac{k_3}{k_2}\right){\rm e}^{-k/k_2} \tag{137}$$

for $k \gtrsim k_3$, where $k_3 = (\mu/4\,\hat{\nu}_{\theta\,i})^{1/2}$. In both regimes IIIa and IIIb,

$$J_p = -\left[I_2 - Q\left(\frac{\rho_i}{w}\right)\right]\epsilon\, v_{\theta\,i}\,(1-v_{\theta\,i}), \tag{138}$$

$$J_g = -I_3\,\alpha_c\,(1+\tau), \tag{139}$$

$$J_b = I_3\,\alpha_n\,\epsilon\,\nu_{\theta\,e}\,\tau_e\,(v_{\theta\,i}+\tau\,v_{\theta\,e}), \tag{140}$$

$$J_s = -4\,(\hat{\nu}_{\perp\,i}\,\mu)^{1/2}\,(v_{\theta\,i}-v). \tag{141}$$

Finally, Regime IV corresponds to $\hat{\nu}_{\theta\,i} \ll \hat{\nu}_{\perp\,i}$ and $\mu \gg \hat{\nu}_{\theta\,i}$. In this regime,

$$Y(k) \simeq \frac{1}{\mathcal{C}}\left(1 - \frac{\hat{\nu}_{\theta\,i}}{\hat{\nu}_{\perp\,i}}\,{\rm e}^{-k/k_3}\right). \tag{142}$$

It follows that

$$J_p = -\left[I_2 - Q\left(\frac{\rho_i}{w}\right)\right]\epsilon\, v\,(1-v), \tag{143}$$

$$J_g = -I_3\,\alpha_c\,(1+\tau), \tag{144}$$

$$J_b = I_3\,\alpha_n\,\epsilon\,\nu_{\theta\,e}\,\tau_e\,(v_{\theta\,i}+\tau\,v_{\theta\,e}), \tag{145}$$

$$J_s = -4\,(\hat{\nu}_{\theta\,i}\,\mu)^{1/2}\,(v_{\theta\,i}-v). \tag{146}$$

J. Freely Rotating Magnetic Islands

Consider a freely rotating magnetic island chain: that is, a chain which is not interacting with a static, resonant, external, magnetic perturbation. This implies that $w_v = 0$ in Eq. (50). Hence, because we have already assumed that $d^2\phi_p/dt^2 = 0$ (i.e., the island is rotating steadily), we conclude that $J_s = 0$. In other words, there is zero local drag torque acting on a freely rotating island chain.

According to the analysis in Sect. III I, a freely rotating island chain is characterized by

$$v = v_{\theta\,i} = 1 + \lambda_{\theta\,i}\left(\frac{\eta_i}{1+\eta_i}\right) = \frac{1-0.172\,\eta_i}{1+\eta_i}, \tag{147}$$

where use has been made of Eq. (28). We conclude that the phase-velocity of a freely rotating chain is solely determined by the neoclassical ion poloidal velocity [which is parameterized by $v_{\theta\,i}$—see Eq. (35)]. Moreover, the phase-velocity lies between the unperturbed local perpendicular guiding-center fluid velocity and the unperturbed local perpendicular ion fluid velocity (i.e., $0 < v < 1$, as is seen experimentally[46]) provided that $0 < \eta_i < 5.81$. On the other hand, if $\eta_i > 5.81$ then the chain rotates in the local electron diamagnetic direction (i.e., $v < 0$).

The analysis in Sect. III I implies that

$$J_p = -\left[1.38 - Q\left(\frac{\rho_i}{w}\right)\right]\epsilon\, v_{\theta\,i}\,(1-v_{\theta\,i}) = -\left[1.38 - Q\left(\frac{\rho_i}{w}\right)\right]\epsilon\, v\,(1-v), \tag{148}$$

where use has been made of Eq. (147). Now, it is clear from Fig. 1 that $1.38 - Q(\rho_i/w) > 0$ unless $\rho_i/w \gtrsim 0.3$. However, $\rho_i/w \gtrsim 0.3$ then is not consistent with an ion-branch magnetic island chain characterized by $w \gg \rho_{\theta\,i}$. We conclude that the perturbed ion polarization current has a stabilizing effect on the island chain (i.e., $J_p < 0$) provided that the chain's phase-velocity lies between the unperturbed local perpendicular guiding-center fluid velocity and the unperturbed local perpendicular ion fluid velocity (i.e., $0 < v < 1$). As we have just seen, this is the case as long as $0 < \eta_i < 5.81$. On the other hand, if $\eta_i > 5.81$ then $v < 0$, and the polarization term becomes destabilizing.

The analysis in Sect. III I yields

$$J_g = -1.58\,\alpha_c\,(1+\tau). \tag{149}$$

In other words, magnetic field-line curvature has a stabilizing effect on the island chain (i.e., $J_g < 0$).

Finally, the analysis in Sect. III I implies that

$$J_b = 3.85\,\epsilon_s^{1/2}\,\alpha_n\left[\frac{(1-0.172\,\eta_i)\,T_{i\,0}+(1+0.283\,\eta_e)\,T_{e\,0}}{(1+\eta_i)\,T_{i\,0}}\right], \tag{150}$$

where we have made use of Eqs. (28) and (30), as well as the standard neoclassical result $\epsilon\,\nu_{\theta\,e}\,\tau_e = 1.67\,f_t$, where $f_t \simeq 1.46\,\epsilon_s^{1/2}$ is the faction of trapped particles at the rational surface.[42,43] It follows that the perturbed bootstrap current is destabilizing (i.e., $J_b > 0$) provided that

$$\eta_i < 5.81\left[1+(1+0.283\,\eta_e)\,\frac{T_{e\,0}}{T_{i\,0}}\right]. \tag{151}$$

K. Locked Magnetic Islands

Consider a locked magnetic island chain: that is, an island chain which is interacting with a static, resonant, external magnetic perturbation whose amplitude is sufficient to reduce the phase-velocity of the island to zero in the laboratory frame. This implies that $v_p = 0$.

It follows from Eqs. (29) and (31) that

$$v = v_{\perp\,i} = 1 + \lambda_{\perp\,i}\left(\frac{\eta_i}{1+\eta_i}\right) = \frac{1-1.367\,\eta_i}{1+\eta_i}. \tag{152}$$

We conclude that, in the local plasma frame, the phase-velocity of a locked magnetic island chain is solely determined by the neoclassical ion perpendicular velocity [which is parameterized by $v_{\perp\,i}$—see Eq. (36)]. Moreover, the phase-velocity lies between the local perpendicular guiding-center fluid velocity and the local perpendicular ion fluid velocity (i.e., $0 < v < 1$) provided that $0 < \eta_i < 0.73$. On the other hand, if $\eta_i > 0.73$ then the chain rotates in the electron diamagnetic direction (i.e., $v < 0$) in the local plasma frame.

The analysis of Sect. III I reveals that the expressions for J_g and J_b are the same for both locked and freely rotating island chains. In other words, magnetic field-line curvature and the perturbed bootstrap current have the same effect on the stability of a locked island chain as they have on that of a corresponding freely rotating chain. On the other hand, the expression for J_p is [cf., Eq. (148)]

$$J_p = -\left[1.38 - Q\left(\frac{\rho_i}{w}\right)\right]\epsilon\, v_{\theta\,i}\,(1 - v_{\theta\,i}) \tag{153}$$

if $\hat{\nu}_{\theta\,i} \gg \hat{\nu}_{\perp\,i}$, and

$$J_p = -\left[1.38 - Q\left(\frac{\rho_i}{w}\right)\right]\epsilon\, v_{\perp\,i}\,(1 - v_{\perp\,i}) = -\left[1.38 - Q\left(\frac{\rho_i}{w}\right)\right]\epsilon\, v\,(1 - v) \tag{154}$$

if $\hat{\nu}_{\theta\,i} \ll \hat{\nu}_{\perp\,i}$. Here, use has been made of Eq. (152). It follows that the perturbed ion polarization current has the same effect on the stability of a locked island chain as it has on that of a corresponding freely rotating chain when the neoclassical ion poloidal flow damping rate greatly exceeds the neoclassical ion perpendicular flow damping rate (i.e., $\hat{\nu}_{\theta\,i} \gg \hat{\nu}_{\perp\,i}$). On the other hand, if the neoclassical ion perpendicular flow damping rate greatly exceeds the neoclassical ion poloidal flow damping rate (i.e., $\hat{\nu}_{\perp\,i} \gg \hat{\nu}_{\theta\,i}$) then the polarization current is stabilizing when $0 < \eta_i < 0.73$, and destabilizing otherwise. In the latter case, if $0.73 < \eta_i < 5.81$ then the polarization current has a stabilizing effect on a freely rotating island chain, but a destabilizing effect on a corresponding locked chain.

IV. STRONG NEOCLASSICAL ION POLOIDAL FLOW DAMPING REGIME

A. Alternative Field Equations

In the so-called strong neoclassical ion poloidal flow damping regime, it is helpful to write the field equations (11)–(15) in the alternative form

$$0 = [\phi + \tau\, N, \psi] + \beta\,\eta\, J$$

$$+\,\alpha_n^{-1}\,\hat{\nu}_{\theta\,e}\left[\alpha_n^{-1}\,J+V-\partial_X(\phi+\tau\,v_{\theta\,e}\,N)-v_{\theta\,i}-\tau\,v_{\theta\,e}\right], \tag{155}$$

$$0=[\phi,N]-\rho\,[\alpha_n\,V+J,\psi]-\alpha_c\,\rho\,[\phi+\tau\,N,X]+D\,\partial_X^2 N, \tag{156}$$

$$0=\delta\,[\phi,V]-\delta\,\alpha_n\,(1+\tau)\,[N,\psi]+\delta\,\mu\,\partial_X^2 V-\delta\,\hat{\nu}_{\theta\,i}\,[V-\partial_X(\phi-v_{\theta\,i}\,N)]\,, \tag{157}$$

$$0=[J,\psi]+\alpha_c\,(1+\tau)\,[N,X]+\partial_X H, \tag{158}$$

$$J=\beta^{-1}\left(\partial_X^2\psi-1\right), \tag{159}$$

where

$$H(X,\zeta)=\epsilon\,[\phi-N,\partial_X\phi]+[\phi,V]-\alpha_n\,(1+\tau)\,[N,\psi]+\mu\,\partial_X^2[V+\epsilon\,\partial_X(\phi-N)]$$
$$+\,\hat{\nu}_{\perp\,i}\,[-\partial_X(\phi-v\,N)]\,. \tag{160}$$

B. Ordering Scheme

The ordering scheme adopted in the strong neoclassical ion poloidal flow damping regime is: [35,36]

$\Delta^0\,\delta^{-1}$: $\hat{\nu}_{\theta\,i}$,

$\Delta^0\,\delta^0$ $v_{\theta\,i}$, $v_{\theta\,e}$, v, τ, α_n, α_c

$\Delta^0\,\delta^1$: ϵ, ρ, β,

$\Delta^1\,\delta^0$: $\hat{\nu}_{\perp\,i}$, η, D, μ,

$\Delta^1\,\delta^1$: $\hat{\nu}_{\theta\,e}$.

This ordering scheme is suitable for a constant-ψ (i.e., $\beta \ll 1$) magnetic island chain whose radial width is much larger than the ion poloidal gyroradius (i.e., $\rho \ll 1$), and which is embedded in a large aspect-ratio (i.e., $\epsilon \ll 1$), high-temperature (i.e., η, D, $\mu \lll 1$) tokamak plasma equilibrium. The defining feature of the strong neoclassical ion poloidal flow damping regime is that the ion poloidal flow damping rate is sufficiently large that the neoclassical ion stress tensor is the dominant term in the ion parallel equation of motion.

C. Order $\Delta^0\,\delta^0$

To order $\Delta^0\,\delta^0$, Eqs. (155)–(160) yield

$$0 = [\phi_{0,0} + \tau\, N_{0,0}, \psi_{0,0}], \tag{161}$$

$$0 = [\phi_{0,0}, N_{0,0}], \tag{162}$$

$$0 = -\,\hat{\nu}_{\theta\,i}\,[V_{0,0} - \partial_X(\phi_{0,0} - v_{\theta\,i}\,N_{0,0})]\,, \tag{163}$$

$$0 = [J_{0,0}, \psi_{0,0}] + \alpha_c\,(1+\tau)\,[N_{0,0}, X] + \partial_X\{[\phi_{0,0}, V_{0,0}] - \alpha_n\,(1+\tau)\,[N_{0,0}, \psi_{0,0}]\}\,, \tag{164}$$

$$\partial_X^2\psi_{0,0} = 1. \tag{165}$$

Equations (40), (45), and (165) give Eq. (58). Equations (41), (42), (161), and (162) lead to Eqs. (59) and (60). Defining $M(\Omega)$ and $L(\Omega)$ in accordance with Eqs. (61) and (62), Eqs. (41) and (42) yield the boundary conditions (63) and (64). As before, $M = L = 0$ inside the separatrix. Equations (59)–(62) and (163) give

$$V_{0,0} = -|X|\,(M - v_{\theta\,i}\,L). \tag{166}$$

According to Eqs. (63) and (64), this expression automatically satisfies the boundary condition (43).

Equations (59)–(62), (164), and (166) yield

$$[J_{0,0}, \Omega] = \left[\frac{1}{2}\,d_\Omega\{(M - v_{\theta\,i}\,L)\,M\}\,X^{\,2} - \alpha_c\,(1+\tau)\,L\,|X|, \Omega\right]. \tag{167}$$

It follows that

$$J_{0,0} = \frac{1}{2}\,d_\Omega[(M - v_{\theta\,i}\,L)\,M]\,\widetilde{X^{\,2}} - \alpha_c\,(1+\tau)\,L\,\widetilde{|X|} + \bar{J}(\Omega), \tag{168}$$

where $\bar{J}(\Omega)$ is an arbitrary flux function. However, the lowest-order flux-surface average of Eq. (155) implies that

$$\bar{J}(\Omega) = \alpha_n\left(\frac{\epsilon\,\nu_{\theta\,e}\,\tau_e}{1+\epsilon\,\nu_{\theta\,e}\,\tau_e}\right)(v_{\theta\,i} + \tau\,v_{\theta\,e})\left(1 - \frac{L}{\langle 1\rangle}\right), \tag{169}$$

where use has been made of Eqs. (59)–(62), and (166).

Finally, it is easily demonstrated that

$$X\,[J_{0,0},\mathit{\Omega}] = \frac{1}{6}\,[X^{\,3},(M-v_{\theta\,i}\,L)\,M] + \frac{1}{2}\,\alpha_c\,(1+\tau)\,[s\,X^{\,2},N_0], \tag{170}$$

which implies that

$$\langle X\,[J_{0,0},\mathit{\Omega}]\rangle = 0. \tag{171}$$

In other words, $J_{0,0}$ does not contribute to the torque integral, J_s [see Eq. (52)]. Thus, in order to calculate J_s, and, hence, to determine the phase-velocity of a freely rotating island chain, we must expand to higher order.

D. Order $\Delta^1\,\delta^{\,0}$

To order $\Delta^1\,\delta^{\,0}$, Eqs. (58)–(60), and (155)–(160) yield

$$0 = [\phi_{1,0}+\tau\,N_{1,0},\mathit{\Omega}] + s\,[\phi_0+\tau\,N_0,\psi_{1,0}], \tag{172}$$

$$0 = s\,[\phi_{1,0},N_0] + s\,[\phi_0,N_{1,0}] + s\,D\,\partial_X^2 N_0, \tag{173}$$

$$0 = -\hat{\nu}_{\theta\,i}\,[V_{1,0}-\partial_X(\phi_{1,0}-v_{\theta\,i}\,N_{1,0})]\,, \tag{174}$$

$$0 = [J_{1,0},\mathit{\Omega}] + [J_{0,0},\psi_{1,0}] + \alpha_c\,(1+\tau)\,[N_{1,0},X] + \partial_X H_{1,0}, \tag{175}$$

$$\partial_X^2\psi_{1,0} = 0, \tag{176}$$

where

$$H_{1,0} = [\phi_{1,0},V_{0,0}] + s\,[\phi_0,V_{1,0}] - \alpha_n\,(1+\tau)\,s\,[N_0,\psi_{1,0}] - \alpha_n\,(1+\tau)\,[N_{1,0},\mathit{\Omega}]$$
$$+\,\mu\,\partial_X^2 V_{0,0} + \hat{\nu}_{\perp\,i}\,[-s\,\partial_X(\phi_0-v\,N_0)]\,. \tag{177}$$

It follows from Eq. (176) that

$$\psi_{1,0} = 0, \tag{178}$$

from Eq. (172) that

$$\phi_{1,0} = -\tau\,N_{1,0}, \tag{179}$$

from Eq. (174) that

$$V_{1,0} = -(v_{\theta\,i}+\tau)\,\partial_X N_{1,0}, \tag{180}$$

from Eq. (173) that

$$[N_{1,0}, \Omega] = D\left(\frac{X^{2}\, d_\Omega L + L}{M+\tau\, L}\right), \tag{181}$$

and from Eqs. (175) and (177) that

$$[J_{1,0}, \Omega] = -\alpha_c\,(1+\tau)\,[N_{1,0}, X] - \partial_X H_{1,0}, \tag{182}$$

where

$$H_{1,0} = \tau\,[N_{1,0}, |X|\,(M - v_{\theta\,i}\,L)] - (v_{\theta\,i}+\tau)\,M\,[|X|\,d_\Omega N_{1,0}, \Omega] - \alpha_n\,(1+\tau)\,[N_{1,0}, \Omega]$$
$$- \mu\left[|X|^3\, d_\Omega^2(M - v_{\theta\,i}\,L) + 3\,|X|\,d_\Omega(M - v_{\theta\,i}\,L)\right] + \hat{\nu}_{\perp\,i}\,|X|\,(M - v\,L). \tag{183}$$

Here, use has been made of Eqs. (58)–(62), and Eq. (166).

As before, the flux-surface average of Eq. (181) yields Eq. (85). Equation (181) then reduces to

$$[N_{1,0}, \Omega] = \left(\frac{D\, d_\Omega L}{M+\tau\, L}\right)\widetilde{X^{2}}. \tag{184}$$

It is clear that $N_{1,0} = 0$ inside the island separatrix (because $L = 0$ there). Hence, we conclude that $H_{1,0} = 0$ inside the separatrix (because $M = L = N_{1,0}$ there).

The flux-surface average of Eq. (182) gives

$$d_\Omega\{\langle |X|\,H_{1,0}\rangle + \alpha_c\,(1+\tau)\,\langle |X|\,[N_{1,0}, \Omega]\rangle\} = 0, \tag{185}$$

which can be integrated to give

$$\langle |X|\,H_{1,0}\rangle + \alpha_c\,(1+\tau)\,\langle |X|\,[N_{1,0}, \Omega]\rangle = 0. \tag{186}$$

Now, it can be demonstrated that [35]

$$\langle |X|^{j}\,H_{1,0}\rangle = (1+j)^{-1}\,[\tau\,(M - v_{\theta\,i}\,L) - j\,(v_{\theta\,i}+\tau)\,M]\,d_\Omega\langle |X|^{j+1}\,[N_{1,0}, \Omega]\rangle$$
$$+ \tau\, d_\Omega(M - v_{\theta\,i}\,L)\,\langle |X|^{j+1}\,[N_{1,0}, \Omega]\rangle$$
$$+ j\,(v_{\theta\,i}+\tau)\,M\,\langle |X|^{j-1}\,[N_{1,0}, \Omega]\rangle - \alpha_n\,(1+\tau)\,\langle |X|^{j}\,[N_{1,0}, \Omega]\rangle$$
$$- \mu\,\langle |X|^{j+3}\rangle\, d_\Omega^2(M - v_{\theta\,i}\,L) - 3\,\mu\,\langle |X|^{j+1}\rangle\, d_\Omega(M - v_{\theta\,i}\,L)$$
$$+ \hat{\nu}_{\perp\,i}\,\langle |X|^{j+1}\rangle\,(M - v\,L), \tag{187}$$

where

$$\langle |X|^j \, [N_{1,0}, \Omega] \rangle = \left(\frac{D \, d_\Omega L}{M + \tau \, L} \right) \left\langle \widetilde{|X|^j} \, \widetilde{X^2} \right\rangle, \tag{188}$$

and j is a non-negative integer. Hence, Eq. (186) yields

$$\begin{aligned} 0 = {} & d_\Omega \left[\langle X^4 \rangle \, d_\Omega (M - v_{\theta\, i} \, L) + \frac{D}{2\,\mu} \, v_{\theta\, i} \left\langle \widetilde{X^2} \, \widetilde{X^2} \right\rangle d_\Omega L \right] \\ & - \frac{D}{2\,\mu} \left\langle \widetilde{X^2} \, \widetilde{X^2} \right\rangle \left[(v_{\theta\, i} + 2\,\tau) \, d_\Omega M - v_{\theta\, i} \, \tau \, d_\Omega L \right] \frac{d_\Omega L}{M + \tau \, L} \\ & + \frac{D}{\mu} \, (\alpha_n - \alpha_c) \, (1 + \tau) \left\langle \widetilde{|X|} \, \widetilde{X^2} \right\rangle \frac{d_\Omega L}{M + \tau \, L} \\ & - \frac{\hat{\nu}_{\perp\, i}}{D} \, \langle X^2 \rangle \, (M - v \, L). \end{aligned} \tag{189}$$

E. Evaluation of J_c

According to Eqs. (85), (168), and (169),

$$\begin{aligned} J_{0,0} = {} & \frac{1}{2} \, d_\Omega \left[M \left(M - \frac{v_{\theta\, i}}{\langle X^2 \rangle} \right) \right] \widetilde{X^2} - \alpha_c \, (1 + \tau) \, \frac{\widetilde{|X|}}{\langle X^2 \rangle} \\ & + \alpha_n \left(\frac{\epsilon \, \nu_{\theta\, e} \, \tau_e}{1 + \epsilon \, \nu_{\theta\, e} \, \tau_e} \right) (v_{\theta\, i} + \tau \, v_{\theta\, e}) \left(1 - \frac{1}{\langle 1 \rangle \langle X^2 \rangle} \right) \end{aligned} \tag{190}$$

for $\Omega \geq 1$, and

$$J_{0,0} = \alpha_n \left(\frac{\epsilon \, \nu_{\theta\, e} \, \tau_e}{1 + \epsilon \, \nu_{\theta\, e} \, \tau_e} \right) (v_{\theta\, i} + \tau \, v_{\theta\, e}) \tag{191}$$

for $-1 \leq \Omega < 1$. Thus, it follows from Eq. (51) that $J_c = J_p + J_g + J_b$, where

$$J_p = \int_{1-}^{\infty} d_\Omega \left[M \left(M - \frac{v_{\theta\, i}}{\langle X^2 \rangle} \right) \right] \langle \widetilde{X^2} \, \widetilde{X^2} \rangle \, d\Omega \tag{192}$$

parameterizes the effect of the perturbed ion polarization current on island stability, whereas

$$J_g = -\alpha_c \, (1 + \tau) \int_1^\infty 2 \, \frac{\langle \widetilde{|X|} \, \widetilde{X^2} \rangle}{\langle X^2 \rangle} \, d\Omega = -I_3 \, \alpha_c \, (1 + \tau) \tag{193}$$

(see the Appendix for the definition of $I_3 = 1.58$) parameterizes the effect of magnetic field-line curvature on island stability, and, finally,

$$\begin{aligned} J_b & = -\alpha_n \left(\frac{\epsilon \, \nu_{\theta\, e} \, \tau_e}{1 + \epsilon \, \nu_{\theta\, e} \, \tau_e} \right) (v_{\theta\, i} + v_{\theta\, e}) \int_1^\infty 2 \, \frac{\langle \widetilde{|X|} \, \widetilde{X^2} \rangle}{\langle X^2 \rangle} \, d\Omega \\ & = -I_3 \, \alpha_n \left(\frac{\epsilon \, \nu_{\theta\, e} \, \tau_e}{1 + \epsilon \, \nu_{\theta\, e} \, \tau_e} \right) (v_{\theta\, i} + v_{\theta\, e}) \end{aligned} \tag{194}$$

parameterizes the effect of the perturbed bootstrap current on island stability. It can be seen, by comparison with the analysis of Sect. III I, that J_g has the same form in both the weak and the strong neoclassical ion flow damping regimes, whereas J_b has very similar forms in the two regimes (in fact, the forms are identical if $\epsilon\,\nu_{\theta\,e}\,\tau_e = 1.67\,f_t = 2.44\,\epsilon_s^{1/2} \ll 1$).

F. Evaluation of J_s

Multiplying Eq. (182) by X, and flux-surface averaging, we obtain

$$\langle X\,[J_{0,1},\mathit{\Omega}]\rangle = -d_{\mathit{\Omega}}\Big\{\langle X^{\,2}\,H_{1,0}\rangle - \frac{\alpha_c}{2}\,(1+\tau)\,\langle X^{\,2}\,[N_{1,0},\mathit{\Omega}]\rangle\Big\} + \langle H_{1,0}\rangle, \tag{195}$$

which can be integrated to give

$$\int_{-1}^{\infty}\langle X\,[J_{0,1},\mathit{\Omega}]\rangle\,d\mathit{\Omega} = \int_{1}^{\infty}\langle H_{1,0}\rangle\,d\mathit{\Omega}, \tag{196}$$

because $H_{1,0}=0$ inside the separatrix. Making use of Eqs. (52), (187), and (188), we obtain

$$J_s = -\hat{\nu}_{\perp\,i}\int_1^{\infty} 4\,(M - v\,L)\,d\mathit{\Omega}. \tag{197}$$

G. Separatrix Boundary Layer

The flux-surface functions $M(\mathit{\Omega})$ and $L(\mathit{\Omega})$ are both zero inside, and non-zero just outside, the magnetic separatrix. Retaining selected higher-order terms (containing radial derivatives) in Eqs. (173) and (175), we find that

$$(M+\tau\,L)\,[N_{1,0},\mathit{\Omega}] - s\,\rho\,[J_{1,0},\mathit{\Omega}] \simeq D\left(X^{\,2}\,d_{\mathit{\Omega}}L + L\right), \tag{198}$$

$$s\,[J_{1,0},\mathit{\Omega}] \simeq -\partial_X[\phi_0,V_{1,0}] = (v_{\theta\,i}+\tau)\,M\,\partial_X^2[N_{1,0},\mathit{\Omega}], \tag{199}$$

so that Eq. (181) generalizes to give

$$\left\{(M+\tau\,L) - (v_{\theta\,i}+\tau)\,M\,\rho\,\partial_X^2\right\}[N_{1,0},\mathit{\Omega}] \simeq D\left(X^{\,2}\,d_{\mathit{\Omega}}L + L\right), \tag{200}$$

which suggests that the apparent discontinuities in the functions $M(\mathit{\Omega})$ and $L(\mathit{\Omega})$ are resolved in a thin boundary layer of (unnormalized) width $(\rho)^{1/2}\,w = \rho_{\theta\,i}$ on the island separatrix.

Inside the boundary layer, Eq. (189) reduces to

$$0 \simeq d_y^2\left(M - v_{\theta\,i}\,L + \frac{D}{2\,\mu}\,v_{\theta\,i}\,L\right) - \frac{D}{2\,\mu}\,d_y\{(v_{\theta\,i}+2\,\tau)\,M - v_{\theta\,i}\,\tau\,L\}\,\frac{d_y L}{M+\tau\,L}, \tag{201}$$

where $y = (\Omega-1)/(\rho_{\theta\,i}/w)$, and $d_y \equiv d/dy$. Note that $d_\Omega \sim \mathcal{O}(w/\rho_{\theta\,i})\,d_y \gg 1$. Here, we have made use of the fact that $\langle X^4\rangle = \langle \widetilde{X^2}\,\widetilde{X^2}\rangle$ close to the separatrix. Let us assume that $M = v_{\theta\,i}\,(1-v_0)\,L$ within the layer, where v_0 is a constant. It follows that

$$0 \simeq d_y(L\,d_y L) - \left[\frac{D}{2\,\mu}\,\frac{(1+2\,\hat{\tau})\,(1-v_0)-\hat{\tau}}{(1-v_0+\hat{\tau})\,(v_0-D/2\,\mu)} - 1\right](d_y L)^2, \tag{202}$$

where $\hat{\tau} = \tau/v_{\theta\,i}$. Integrating across the layer from just inside the separatrix (i.e., $y \to -\infty$, where $L = 0$) to just outside the separatrix [i.e., $y \to \infty$, where $d_y L \sim \mathcal{O}(\rho_{\theta\,i}/w) \ll 1$, because $d_\Omega L \sim \mathcal{O}(1)$], we obtain

$$\left[\frac{D}{2\,\mu}\,\frac{(1+2\,\hat{\tau})\,(1-v_0)-\hat{\tau}}{(1-v_0+\hat{\tau})\,(v_0-D/2\,\mu)} - 1\right]\int_{-\infty}^{\infty}(d_y L)^2\,dy \ll 1. \tag{203}$$

Now, the integral in the previous expression is positive definite, and also of order unity. Thus, the only way in which Eq. (203) can be satisfied is if

$$\frac{D}{2\,\mu}\,\frac{(1+2\,\hat{\tau})\,(1-v_0)-\hat{\tau}}{(1-v_0+\hat{\tau})\,(v_0-D/2\,\mu)} = 1, \tag{204}$$

or

$$v_0^2 - (1+\hat{\tau})\left(1+\frac{D}{\mu}\right)v_0 + (1+\hat{\tau})\,\frac{D}{\mu} = 0, \tag{205}$$

which implies that

$$v_0 = \left(\frac{1+\hat{\tau}}{2}\right)\left(1+\frac{D}{\mu} - \left[1 - 2\,\frac{D}{\mu}\left(\frac{1-\hat{\tau}}{1+\hat{\tau}}\right) + \left(\frac{D}{\mu}\right)^2\right]^{1/2}\right). \tag{206}$$

Here, we have chosen the root of the quadratic equation (205) that corresponds to the obvious physical solution $v_0 = 0$ when $D/\mu = 0$.[15] Note that $0 \leq v_0 \leq 1$.

H. Transformed Equations

Equation (189) reduces to

$$\begin{aligned} 0 = d_k\Bigg[2\,k^{\,2}\,\mathcal{E}\,d_k M + v_{\theta\,i}\left(1-\frac{D}{2\,\mu}\right)\left(\frac{\mathcal{E}\,\mathcal{A}}{\mathcal{C}^{\,2}}-1\right)\Bigg] \\ +\frac{D}{2\,\mu}\left(\frac{\mathcal{E}\,\mathcal{A}}{\mathcal{C}^{\,2}}-1\right)\left[\frac{(v_{\theta\,i}+2\,\tau)\,2\,k\,\mathcal{C}\,d_k M + v_{\theta\,i}\,\tau\,\mathcal{A}/k\,\mathcal{C}}{2\,k\,\mathcal{C}\,M+\tau}\right] \\ -\frac{D}{\mu}\,(\alpha_n-\alpha_c)\,(1+\tau)\left(\frac{\mathcal{D}\,\mathcal{A}}{\mathcal{C}}-1\right)\left(\frac{4\,k}{2\,k\,\mathcal{C}\,M+\tau}\right) \\ -\frac{\hat{\nu}_{\perp\,i}}{\mu}\,4\,k\,(2\,k\,\mathcal{C}\,M-v), \end{aligned} \tag{207}$$

where $k=[(1+\Omega)/2]^{\,1/2}$ and $\mathcal{A}(k)$, $\mathcal{C}(k)$, $\mathcal{D}(k)$, and $\mathcal{E}(k)$ are defined in the Appendix. It follows from Eq. (63), and the analysis of Sect. IV G, that

$$M(1) = v_{\theta\,i}\,(1-v_0)\,\frac{\pi}{4}, \tag{208}$$

$$M(k\rightarrow\infty) = \frac{v}{2\,k}. \tag{209}$$

Furthermore, Equations (192) and (197) yield

$$J_p = -\left[\frac{2\pi}{3}-Q\left(\frac{\rho_{\theta\,i}}{w}\right)\right]v_{\theta\,i}^2\,v_0\,(1-v_0) + \int_{1+}^{\infty} d_k\Big[M\left(M-\frac{v_{\theta\,i}}{2\,k\,\mathcal{C}}\right)\Big]\,8\left(\mathcal{E}-\frac{\mathcal{C}^{\,2}}{\mathcal{A}}\right)k^{\,3}\,dk, \tag{210}$$

$$J_s = -\hat{\nu}_{\perp\,i}\int_1^\infty 16\left(M\,k-\frac{v}{2\,\mathcal{C}}\right)dk, \tag{211}$$

respectively. Here, we have evaluated the contribution to the polarization integral emanating from the boundary layer on the magnetic separatrix [i.e., the first term on the right-hand side of Eq. (210)] according to the method set out in Sect. III H, taking into account the fact that that the (unnormalized) thickness of the layer is $\rho_{\theta\,i}$. It remains to solve Eq. (207), subject to the boundary conditions (208) and (209), and then to evaluate the integrals (210) and (211). This task, which involves the elimination of an unphysical solution that varies as $M\sim\exp[2\,(\hat{\nu}_{\perp\,i}/\mu)^{1/2}\,k]$ at large k, can be performed analytically in the strong ion perpendicular flow damping regime, $\hat{\nu}_{\perp\,i}\gg\mu$, but must, otherwise, be performed numerically.

I. Weak Neoclassical Ion Perpendicular Flow Damping Regime

Suppose that $\hat{\nu}_{\perp\,i}/\mu \ll 1$. In the limit $k \gg 1$, Eq. (207) reduces to

$$d_k(2\,k^{\,2}\,d_k M) - \frac{\hat{\nu}_{\perp\,i}}{\mu}\,4\,k\,(2\,k\,M - v) = 0. \tag{212}$$

The solution is

$$M(k) = \frac{v + (v_{\theta\,i}\,v_f - v)\,{\rm e}^{-2\,(\hat{\nu}_{\perp\,i}/\mu)^{1/2}\,k}}{2\,k}, \tag{213}$$

where use has been made of Eq. (209). Here, v_f is an arbitrary constant.

It follows from Eq. (211) that

$$J_s = -4\,(\hat{\nu}_{\perp\,i}\,\mu)^{1/2}\,(v_{\theta\,i}\,v_f - v). \tag{214}$$

Hence, v_f determines the phase-velocity of a freely rotating magnetic island chain. (See Sect. III J.)

In the region $1 \leq k \ll (\mu/\hat{\nu}_{\perp\,i})^{\,1/2}$, Eq. (207) reduces to

$$\begin{aligned} 0 = d_k&\left[2\,k^{\,2}\,\mathcal{E}\,d_k\widehat{M} + \left(1 - \frac{D}{2\,\mu}\right)\frac{\mathcal{E}\,\mathcal{A}}{\mathcal{C}^{\,2}}\right] \\ &+ \frac{D}{2\,\mu}\left(\frac{\mathcal{E}\,\mathcal{A}}{\mathcal{C}^{\,2}} - 1\right)\left[\frac{(1+2\,\hat{\tau})\,2\,k\,\mathcal{C}\,d_k\widehat{M} + \hat{\tau}\,\mathcal{A}/k\,\mathcal{C}}{2\,k\,\mathcal{C}\,\widehat{M} + \hat{\tau}}\right] \\ &- \frac{D}{\mu}\,\Lambda\left(\frac{\mathcal{D}\,\mathcal{A}}{\mathcal{C}} - 1\right)\left[\frac{4\,k\,(1+\hat{\tau})}{2\,k\,\mathcal{C}\,\widehat{M} + \hat{\tau}}\right], \end{aligned} \tag{215}$$

where

$$\widehat{M}(k) = \frac{M(k)}{v_{\theta\,i}}, \tag{216}$$

$$\Lambda = \frac{(\alpha_n - \alpha_c)\,(1+\tau)}{v_{\theta\,i}\,(v_{\theta\,i} + \tau)}. \tag{217}$$

Recall that $\hat{\tau} = \tau/v_{\theta\,i}$. Equation (215) must be solved subject to the boundary conditions

$$\widehat{M}(1) = (1 - v_0)\,\frac{\pi}{4}, \tag{218}$$

$$\widehat{M}(k\rightarrow\infty) = \frac{v_f}{2\,k}, \tag{219}$$

where use has been made of Eqs. (208) and (213). Finally,

$$J_p = v_{\theta\,i}^2\,\hat{J}_p, \tag{220}$$

where

$$\hat{J}_p = -\left[\frac{2\pi}{3} - Q\left(\frac{\rho_{\theta\,i}}{w}\right)\right] v_0\,(1-v_0) + \int_{1+}^{\infty} d_k\left[\widehat{M}\left(\widehat{M} - \frac{1}{2\,k\,\mathcal{C}}\right)\right] 8\left(\mathcal{E} - \frac{\mathcal{C}^{\,2}}{\mathcal{A}}\right) k^{\,3}\,dk. \tag{221}$$

It is clear that $v_f = v_f(D/\mu, \mathit{\Lambda}, \hat{\tau})$ and $\hat{J}_p = \hat{J}_p(D/\mu, \mathit{\Lambda}, \hat{\tau}, \rho_{\theta\,i}/w)$. Figures 3 and 4 show v_f and $\hat{J}_p$, calculated numerically as functions of D/μ, for $\hat{\tau} = 1$, $\rho_{\theta\,i}/w = 10^{-3}$, and $\mathit{\Lambda} = 0.5$, 0.0, and -0.5. Note that $v < 1$, which implies that a freely rotating island chain propagates in the electron diamagnetic direction relative to the unperturbed local ion fluid, and $\hat{J}_p < 0$, which implies that the perturbed polarization current has a stabilizing effect on the chain. Figure 5 shows $\hat{J}_p$, calculated as a function of D/μ, for $\hat{\tau} = 1$, $\mathit{\Lambda} = 0$, and $\rho_{\theta\,i}/w = 10^{-5}$, 10^{-3}, and 10^{-1}. It can be seen that the magnitude of the polarization integral, $\hat{J}_p$, decreases significantly as the relative width of the separatrix boundary layer increases.[26,27] However, the sign of the integral remains negative. This shows that, although the contribution of the separatrix boundary layer to the polarization integral is reduced when the finite width of the layer is taken into account, the layer contribution still remains large enough to determine the sign of the integral. (Note that if the layer contribution were entirely neglected then the integral would be positive.)

J. Strong Neoclassical Ion Perpendicular Flow Damping Regime

Suppose that $\hat{\nu}_{\perp\,i}/\mu \gg 1$. Equations (207)–(209) give

$$M(k) = \frac{v + [v_{\theta\,i}\,(1-v_0) - v]\,\mathrm{e}^{-(6\,\hat{\nu}_{\perp\,i}/\mu)^{1/2}\,(k-1)}}{2\,k\,\mathcal{C}}. \tag{222}$$

It follows from Eqs. (210) and (211) that

$$J_p = -\left[I_2 - Q\left(\frac{\delta_\perp}{w}\right)\right] v\,(v_{\theta\,i} - v) - \left[Q\left(\frac{\delta_\perp}{w}\right) - Q\left(\frac{\rho_{\theta\,i}}{w}\right)\right] v_{\theta\,i}^2\,v_0\,(1-v_0), \tag{223}$$

and

$$J_s = -\sqrt{\frac{8}{3}}\,\pi\,(\hat{\nu}_{\perp\,i}\,\mu)^{1/2}\,[v_{\theta\,i}\,(1-v_0) - v]\,, \tag{224}$$

respectively. Here,

$$\delta_\perp = \left(\frac{q_s}{\epsilon_0}\right)\left(\frac{2}{3}\,\frac{\mu_{\perp\,i}}{n_0\,m_i\,\nu_{\perp\,i}}\right)^{1/2}, \tag{225}$$

and it is assumed that $\rho_{\theta\,i} \ll \delta_\perp \ll w$. Note that we have again evaluated the contributions to the polarization integral emanating from boundary layers on the magnetic separatrix [i.e., both terms on the right-hand side of Eq. (223)] according to the method set out in Sect. III H. Furthermore, $I_2 = 1.38$ is defined in the Appendix.

K. Freely Rotating Magnetic Islands

This subsection, and the following subsection, will concentrate on the strong neoclassical perpendicular flow damping regime, discussed in Sect. IV J, for which we possess an analytic solution.

Consider a freely rotating magnetic island chain. As discussed in Sect. III J, there is zero local drag torque acting on such a chain (i.e., $J_s = 0$). Thus, it follows from Eqs. (28), (206), and (224) that the chain's phase-velocity parameter is given by

$$v = v_{\theta\,i}\,(1 - v_0) = \left(\frac{v_{\theta\,i}+\tau}{2}\right)\left(\frac{v_{\theta\,i}+\tau}{v_{\theta\,i}-\tau} - \frac{D}{\mu} + \left[1 - 2\,\frac{D}{\mu}\left(\frac{v_{\theta\,i}+\tau}{v_{\theta\,i}-\tau}\right) + \left(\frac{D}{\mu}\right)^2\right]^{1/2}\right), \tag{226}$$

where

$$v_{\theta\,i} = \frac{1 - 0.172\,\eta_i}{1+\eta_i}. \tag{227}$$

In the limit $D/\mu \ll 1$, the previous two equations reduce to

$$v \simeq \left(\frac{1 - 0.172\,\eta_i}{1+\eta_i}\right)\left(1 - \frac{D}{\mu}\right), \tag{228}$$

whereas in the opposite limit $\mu/D \ll 1$, we get

$$v \simeq \left(\frac{1 - 0.172\,\eta_i}{1+\eta_i}\right)\frac{\mu}{D}. \tag{229}$$

We conclude that the phase-velocity of a freely rotating magnetic island chain is determined by the neoclassical ion poloidal velocity (which is parameterized by $v_{\theta\,i}$), the ratio of the perpendicular particle and momentum diffusivities (which is parameterized by D/μ), and the electron-ion temperature ratio (which is parameterized by τ). The phase-velocity lies between the unperturbed local perpendicular guiding-center fluid velocity and the unperturbed local perpendicular ion fluid velocity (i.e., $0 < v < 1$, as seen in experiments[46]) provided that $0 < \eta_i < 5.81$. On the other hand, if $\eta_i > 5.81$ then the chain rotates in the electron diamagnetic direction (i.e., $v < 0$).

According to Eq. (223),

$$J_p = -\left[1.38 - Q\left(\frac{\rho_{\theta\,i}}{w}\right)\right] v_{\theta\,i}^2\, v_0\,(1-v_0), \tag{230}$$

where v_0 is specified in Eq. (206). Note that $0 < v_0 < 1$. In the limit $D/\mu \ll 1$, we get

$$J_p \simeq -\left[1.38 - Q\left(\frac{\rho_{\theta\,i}}{w}\right)\right]\left(\frac{1-0.172\,\eta_i}{1+\eta_i}\right)^2 \frac{D}{\mu}, \tag{231}$$

whereas in the opposite limit $\mu/D \ll 1$, we obtain

$$J_p \simeq -\left[1.38 - Q\left(\frac{\rho_{\theta\,i}}{w}\right)\right]\left(\frac{1-0.172\,\eta_i}{1+\eta_i}\right)^2 \frac{\mu}{D}. \tag{232}$$

Assuming that $1.38 > Q(\rho_{\theta\,i}/w)$ (which must be the case, otherwise the width of the separatrix boundary layer would be comparable with that of the island, thus, invalidating our analysis—see Fig. 1), we conclude that the perturbed ion polarization current always has a stabilizing effect on the island chain (i.e. $J_p < 0$). Note, incidentally, that, in the strong neoclassical ion poloidal flow damping regime, J_p is a factor $\epsilon^{-1} = (q_s/\epsilon_s)^2$ larger in magnitude than in the weak neoclassical ion poloidal flow damping regime (see Sect. III J).[29,30]

L. Locked Magnetic Islands

Consider a locked magnetic island chain, which is characterized by $v_p = 0$. It follows from Eqs. (29) and (31) that

$$v = v_{\perp\,i} = 1 + \lambda_{\perp\,i}\left(\frac{\eta_i}{1+\eta_i}\right) = \frac{1-1.367\,\eta_i}{1+\eta_i}, \tag{233}$$

We conclude that, in the local plasma frame, the phase-velocity of a locked magnetic island chain is solely determined by the neoclassical ion perpendicular velocity (which is parameterized by $v_{\perp\,i}$). Moreover, the phase-velocity lies between the local perpendicular guiding-center fluid velocity and the local perpendicular ion fluid velocity (i.e., $0 < v < 1$) provided that $0 < \eta_i < 0.73$. On the other hand, if $\eta_i > 0.73$ then the chain rotates in the electron diamagnetic direction (i.e., $v < 0$) in the local plasma frame.

According to Eqs. (28), (29), (31), and (223),

$$
\begin{aligned}
J_p &= -\left[1.38 - Q\left(\frac{\rho_{\theta\,i}}{w}\right)\right] v_{\perp\,i}\,(v_{\theta\,i} - v_{\perp\,i}) - \left[Q\left(\frac{\delta_\perp}{w}\right) - Q\left(\frac{\rho_{\theta\,i}}{w}\right)\right] v_{\theta\,i}^2\, v_0\,(1 - v_0)\\
&= -1.65\,\frac{\eta_i\,(1 - 1.367\,\eta_i)}{(1+\eta_i)^2}\left[1 - 0.72\,Q\left(\frac{\rho_{\theta\,i}}{w}\right)\right]\\
&\quad - \left(\frac{1 - 0.172\,\eta_i}{1+\eta_i}\right)^2 v_0\,(1 - v_0)\left[Q\left(\frac{\delta_\perp}{w}\right) - Q\left(\frac{\rho_{\theta\,i}}{w}\right)\right].
\end{aligned}
\tag{234}
$$

Figure 6 shows J_p plotted as a function of η_i for various different values of $\delta_\perp/\rho_{\theta\,i}$. It can be seen that the perturbed ion polarization current has a stabilizing effect on a locked magnetic island chain (i.e., $J_p < 0$) when η_i is less than about 0.75, and a destabilizing effect when it exceeds this value.

V. SUMMARY AND CONCLUSIONS

In this paper, we have calculated the effect of the perturbed ion polarization current on the stability of ion-branch, neoclassical tearing modes using an improved, neoclassical, four-field, drift-MHD model. The improvements to the model are described in the Introduction. The calculation involves the self-consistent determination of the pressure and scalar electric potential profiles in the vicinity of the associated magnetic island chain, which allows a determination of the chain's propagation velocity. We have considered two regimes. First, the so-called *weak neoclassical ion poloidal flow damping regime* in which neoclassical ion poloidal flow damping is not strong enough to enhance the magnitude of the polarization current (relative to that found in slab geometry)—see Sect. III. Second, the so-called *strong neoclassical ion poloidal flow damping regime* in which neoclassical ion poloidal flow damping is strong enough to significantly enhance the magnitude of the polarization current—see Sect. IV. In both regimes, we have considered two types of solution. First, *freely rotating* solutions (i.e., island chains that are not interacting with static, resonant, magnetic perturbations)—see Sects. III J and IV K. Second, *locked* solutions (i.e., island chains that have been brought to rest in the laboratory frame via interaction with static, resonant, magnetic perturbations)—see Sects. III K and IV L.

In the weak neoclassical ion poloidal flow damping regime, the island width evolution equation of a freely rotating island chain takes the form

$$0.823\,\tau_R\,\frac{d}{dt}\left(\frac{W}{r_s}\right) = \Delta'\,r_s$$
$$+\,15.41\,\epsilon_s^{1/2}\,\beta_p\left(\frac{L_q}{L_n}\right)\left(\frac{r_s}{W}\right)\left[(1-0.172\,\eta_i)+(1+0.283\,\eta_e)\,\frac{T_e}{T_i}\right]$$
$$-\,12.64\,\beta_p\left(\frac{L_q^2}{L_n\,L_c}\right)\left(\frac{r_s}{W}\right)\left[(1+\eta_i)+(1+\eta_i)\,\frac{T_e}{T_i}\right]$$
$$-\,103.5\,\beta_p\left(\frac{L_q}{L_n}\right)^2\left(\frac{\rho_i}{W}\right)^2\left(\frac{r_s}{W}\right)P\left(\frac{\rho_i}{W}\right)\eta_i\,(1-0.172\,\eta_i), \qquad (235)$$

where

$$P(x) = 1-\frac{4.5}{\ln(4/x)}-\frac{2.2}{\ln^2(4/x)}. \qquad (236)$$

Here, $W = 4\,w$ is the full radial island width, r_s the minor radius of the rational surface, $\epsilon_s = r_s/R_0$ the inverse aspect-ratio, R_0 the major radius, $\tau_R = \mu_0\,r_s^2/\eta_\parallel$ the resistive diffusion timescale, $\eta_\parallel$ the parallel resistivity, Δ' the tearing stability index, $\beta_p = \mu_0\,n_e\,T_i/B_\theta^2$ the poloidal ion beta, $\rho_i = (T_i/m_i)^{1/2}\,(m_i/e\,B_\varphi)$ the ion gyroradius, B_θ the equilibrium poloidal magnetic field-strength, B_φ the equilibrium toroidal magnetic field-strength, m_i the ion mass, e the magnitude of the electron charge, n_e the equilibrium electron number density, T_i the equilibrium ion temperature, T_e the equilibrium electron temperature, L_n the equilibrium density scale-length, L_q the equilibrium safety-factor scale-length, L_c the mean radius of curvature of magnetic field-lines, $\eta_i = d\ln T_i/d\ln n_e$, and $\eta_e = d\ln T_e/d\ln n_e$. All quantities are evaluated at the rational surface. The first term on the right-hand side of the previous equation governs the intrinsic stability of the island chain, the second term parameterizes the effect of the perturbed bootstrap current on island stability, the third term parameterizes the effect of magnetic field-line curvature on island stability, and the final term parameterizes the effect of the perturbed ion polarization current on island stability. It can be seen that the perturbed bootstrap current is destabilizing when $\eta_i < 5.81\,[1+(1+0.283\,\eta_e)\,T_e/T_i]$, magnetic field-line curvature is always stabilizing, and the perturbed ion polarization current is stabilizing provided that $0 < \eta_i < 5.81$.

In the weak neoclassical ion poloidal flow damping regime, the island width evolution equation of a locked island chain takes the form (235) when the neoclassical ion poloidal flow damping rate greatly exceeds the neoclassical ion perpendicular flow damping rate. However, in the opposite limit, the island width evolution equation becomes

$$
\begin{aligned}
0.823\,\tau_R\,\frac{d}{dt}\left(\frac{W}{r_s}\right) &= \Delta'\,r_s \\
&+ 15.41\,\epsilon_s^{1/2}\,\beta_p\left(\frac{L_q}{L_n}\right)\left(\frac{r_s}{W}\right)\left[(1-0.172\,\eta_i)+(1+0.283\,\eta_e)\,\frac{T_e}{T_i}\right] \\
&- 12.64\,\beta_p\left(\frac{L_q^2}{L_n\,L_c}\right)\left(\frac{r_s}{W}\right)\left[(1+\eta_i)+(1+\eta_i)\,\frac{T_e}{T_i}\right] \\
&- 209.3\,\beta_p\left(\frac{L_q}{L_n}\right)^2\left(\frac{\rho_i}{W}\right)^2\left(\frac{r_s}{W}\right)P\left(\frac{\rho_i}{W}\right)\eta_i\,(1-1.367\,\eta_i).
\end{aligned}
\qquad (237)
$$

It can be seen that, in this case, the ion polarization term is modified in such a manner that it is stabilizing when $0 < \eta_i < 0.73$. This result gives rise to the interesting possibility that, when $0.73 < \eta_i < 5.81$, the polarization current can have a stabilizing effect on a freely rotating island chain, but a destabilizing effect on a corresponding locked chain. This may help to explain the common experimental observation that locked magnetic island chains grow to anomalously large widths compared to similar freely rotating chains.

In the strong neoclassical ion poloidal flow damping regime, the island width evolution equation of a freely rotating island chain takes the form

$$
\begin{aligned}
0.823\,\tau_R\,\frac{d}{dt}\left(\frac{W}{r_s}\right) &= \Delta'\,r_s \\
&+ 15.41\,\epsilon_s^{1/2}\,\beta_p\left(\frac{L_q}{L_n}\right)\left(\frac{r_s}{W}\right)\left[(1-0.172\,\eta_i)+(1+0.283\,\eta_e)\,\frac{T_e}{T_i}\right] \\
&- 12.64\,\beta_p\left(\frac{L_q^2}{L_n\,L_c}\right)\left(\frac{r_s}{W}\right)\left[(1+\eta_i)+(1+\eta_i)\,\frac{T_e}{T_i}\right] \\
&- 88.32\,\mathcal{E}\,\beta_p\left(\frac{L_q}{L_n}\right)^2\left(\frac{\rho_i}{W}\right)^2\left(\frac{r_s}{W}\right)P\left(\frac{\rho_{\theta\,i}}{W}\right)(1-0.172\,\eta_i)^2\,v_0\,(1-v_0),
\end{aligned}
\qquad (238)
$$

where

$$
\mathcal{E} = \left(\frac{q_s}{\epsilon_s}\right)^2, \qquad (239)
$$

$$v_0 = \left(\frac{1+\tau_0}{2}\right)\left(1+\frac{D}{\mu}-\left[1-2\,\frac{D}{\mu}\left(\frac{1-\tau_0}{1+\tau_0}\right)+\left(\frac{D}{\mu}\right)^2\right]^{1/2}\right), \tag{240}$$

$$\tau_0 = \left(\frac{1+\eta_e}{1-0.172\,\eta_i}\right)\frac{T_e}{T_i}. \tag{241}$$

Here, D/μ is the ratio of the perpendicular particle diffusivity to the perpendicular ion momentum diffusivity at the rational surface, q_s the safety-factor at the rational surface, and $\rho_{\theta\,i} = (q_s/\epsilon_s)\,\rho_i$ the poloidal ion gyroradius. Note that $0 \leq v_0 \leq 1$. It can be seen, by comparison with Eq. (235), that in the strong neoclassical ion poloidal flow damping regime the ion polarization term is enhanced by a factor $\mathcal{E} = (q_s/\epsilon_s)^2$ compared to that in the weak neoclassical ion poloidal flow damping regime.[29,30] Moreover, the polarization term is always stabilizing (except if $\eta_i = 5.81$, when it is zero). [Note, incidentally, that the expression for the ion polarization term appearing in Eq. (238) only holds in the strong neoclassical ion perpendicular flow damping regime, discussed in Sect. IV J. In the weak neoclassical ion perpendicular flow damping regime, discussed in Sect. IV I, the expression for the polarization term is similar in nature, but much more complicated in form.]

Finally, in the strong neoclassical ion poloidal flow damping regime, the island width evolution equation of a locked island chain takes the form

$$\begin{aligned}0.823\,\tau_R\,\frac{d}{dt}\left(\frac{W}{r_s}\right) = {}& \Delta'\,r_s\\ &+ 15.41\,\epsilon_s^{1/2}\,\beta_p\left(\frac{L_q}{L_n}\right)\left(\frac{r_s}{W}\right)\left[(1-0.172\,\eta_i)+(1+0.283\,\eta_e)\,\frac{T_e}{T_i}\right]\\ &- 12.64\,\beta_p\left(\frac{L_q^2}{L_n\,L_c}\right)\left(\frac{r_s}{W}\right)\left[(1+\eta_i)+(1+\eta_i)\,\frac{T_e}{T_i}\right]\\ &- 105.6\,\mathcal{E}\,\beta_p\left(\frac{L_q}{L_n}\right)^2\left(\frac{\rho_i}{W}\right)^2\left(\frac{r_s}{W}\right)\eta_i\,(1-1.367\,\eta_i) \qquad (242)\\ &- 88.32\,\mathcal{E}\,\beta_p\left(\frac{L_q}{L_n}\right)^2\left(\frac{\rho_i}{W}\right)^2\left(\frac{r_s}{W}\right)\left[P\left(\frac{\delta_\perp}{W}\right)-P\left(\frac{\rho_{\theta\,i}}{W}\right)\right]\\ &\times(1-0.172\,\eta_i)^2\,v_0\,(1-v_0), \qquad (243)\end{aligned}$$

Here, $\delta_\perp = (q_s/\epsilon_s)\,(2\,\mu_{\perp\,i}/3\,n_e\,m_i\,\nu_{\perp\,i})^{1/2}$, where $\mu_{\perp\,i}$ is the ion perpendicular viscosity, and $\nu_{\perp\,i}$ the neoclassical ion perpendicular damping rate. Roughly speaking, in this case, the ion polarization term is modified in such a manner that it is only stabilizing when $0 < \eta_i \lesssim 0.75$. (See Fig. 6). This result again gives rise to the interesting possibility that, when $\eta_i \gtrsim 0.75$, the polarization current can have a stabilizing effect on a freely rotating island chain, but a destabilizing effect on a corresponding locked chain.

According to fluid theory, the enhancement factor of the perturbed ion polarization current in the strong neoclassical ion poloidal flow damping regime is $\mathcal{E} = (q_s/\epsilon_s)^2$.[29,30] However, it should be noted that, according to kinetic theory,[22,49,50] there exists an intermediate neoclassical ion poloidal flow damping regime in which the enhancement factor is reduced to $\epsilon_s^{3/2}\,(q_s/\epsilon_s)^2$. In this intermediate regime, only the contribution of the trapped ions to the polarization current is enhanced. It is possible to crudely incorporate the intermediate flow damping regime into our analysis by writing the enhancement factor in the form[51]

$$\mathcal{E} = \epsilon_s^{3/2}\left(\frac{|v| + \hat{\nu}_i}{|v| + \epsilon_s^{3/2}\,\hat{\nu}_i}\right)\left(\frac{q_s}{\epsilon_s}\right)^2. \tag{244}$$

Here, $\hat{\nu}_i = \nu_i/(\epsilon_s\,k_\theta\,V_{\ast\,i})$, where ν_i is the ion collision frequency, and v is the island phase-velocity parameter defined in Eq. (31). For the case of a freely rotating island chain,

$$v = \left(\frac{1 - 0.172\,\eta_i}{1 + \eta_i}\right) v_0. \tag{245}$$

On the other hand, for the case of a locked island chain,

$$v = \frac{1 - 1.367\,\eta_i}{1 + \eta_i}. \tag{246}$$

For the case of a freely rotating magnetic island chain in the strong neoclassical ion poloidal flow damping regime (which is the regime that is most relevant to experiments), Eqs. (238), (244), and (245) yield the following expression for the threshold island width above which a neoclassical tearing mode grows to large amplitude:

$$W_{\rm crit} = 2.39\,\rho_{b\,i}\left(\frac{L_q}{L_n}\right)^{1/2} P^{1/2}\left(\frac{\rho_{\theta\,i}}{W_{\rm crit}}\right) F(\eta_e, \eta_i, T_e/T_i, D/\mu, L_q/L_c, \hat{\nu}_i, \epsilon_s), \tag{247}$$

where

$$F = \sqrt{G\,H}, \tag{248}$$

$$G = \frac{|1-0.172\,\eta_i|\,v_0+\hat{\nu}_i\,(1+\eta_i)}{|1-0.172\,\eta_i|\,v_0+\epsilon_s^{3/2}\,\hat{\nu}_i\,(1+\eta_i)}, \tag{249}$$

$$H = \frac{(1-0.172\,\eta_i)^{\,2}\,v_0\,(1-v_0)}{(1-0.172\,\eta_i)+(1+0.283\,\eta_e)\,(T_e/T_i)-0.820\,(L_q/\epsilon_s^{1/2}L_c)\,[(1+\eta_i)+(1+\eta_e)\,(T_e/T_i)]}, \tag{250}$$

and $\rho_{b\,i} = \epsilon_s^{1/2}\,(q_s/\epsilon_s)\,\rho_i$ is the ion banana width. Here, v_0 is specified in Eqs. (240) and (241), whereas P is specified in Eq. (236). Figure 7 shows F calculated as a function of η_i for various different values of $\hat{\nu}_i$. In all cases, it can be seen that F falls to zero at $\eta_i = 5.81$. As is clear from Eq. (245), this is the critical value of η_i above which the island rotation, relative to the local guiding center fluid, switches from the ion to the electron diamagnetic direction. Such a switch is likely to trigger a neoclassical tearing mode (because the threshold island width falls to zero as the switch occurs). A reduction in collisionality (i.e., in $\hat{\nu}_i$) is also likely to trigger a neoclassical tearing mode (because the threshold island width is a decreasing function of $\hat{\nu}_i$).

ACKNOWLEDGEMENTS

This research was funded by the U.S. Department of Energy under contract DE-FG02-04ER-54742.

---

[1] J.A. Wesson, *Tokamaks*, 3rd Edition. (Oxford University Press, 2004).

[2] A.H. Boozer, Rev. Mod. Phys. **76**, 1071 (2004).

[3] J.P. Freidberg, *Ideal Magnetohydrodynamics.* (Springer, 1987).

[4] J.A. Wesson, Nucl. Fusion **18**, 87 (1978).

[5] H.P. Furth, J. Killeen, and M.N. Rosenbluth, Phys. Fluids **6**, 459 (1963).

[6] P.H. Rutherford, Phys. Fluids **16**, 1903 (1973).

[7] A. Thyagaraja, Phys. Fluids **24**, 1716 (1981).

[8] D.F. Escande, and M. Ottaviani, Phys. Lett. A **323**, 278 (2004).

[9] R.J. Hastie, F. Militello, and F. Porcelli, Phys. Rev. Lett. **95**, 065001 (2005).

[10] Z. Chang, and J.D. Callen, Nucl. Fusion **30**, 219 (1990).

[11] R.D. Hazeltine, and J.D. Meiss, *Plasma Confinement*. (Dover, 2003).

[12] M. Ottaviani, F. Porcelli, and D. Grasso, Phys. Rev. Lett. **93**, 075001 (2004).

[13] R. Fitzpatrick, F.L. Waelbroeck, and F. Militello, Phys. Plasmas **13**, 122507 (2006).

[14] R. Fitzpatrick, and F.L. Waelbroeck, Phys. Plasmas **14**, 122502 (2007).

[15] R. Fitzpatrick, and F.L. Waelbroeck, Phys. Plasmas **15**, 012502 (2008).

[16] R.J. Bickerton, J.W. Connor, and J.B. Taylor, Nat. Phys. Sci. **229**, 110 (1971).

[17] R. Carrera, R.D. Hazeltine, and M. Kotschenreuther, Phys. Fluids **29**, 899 (1986).

[18] C.C. Hegna, and J.D. Callen, Phys. Fluids B **4**, 1855 (1992).

[19] Z. Chang, J.D. Callen, E.D. Fredrickson, R.V. Budny, C.C. Hegna, K.M. McGuire, M.C. Zarnstorff, and TFTR group, Phys. Rev. Lett. **74**, 4663 (1995).

[20] R. Fitzpatrick, Phys. Plasmas **2**, 825 (1995).

[21] A.I. Smolyakov, Sov. J. Plasma Phys. **15**, 667 (1989).

[22] H.R. Wilson, J.W. Connor, R.J. Hastie, and C.C. Hegna, Phys. Plasmas **3**, 248 (1996).

[23] R. Fitzpatrick, and F.L. Waelbroeck, Phys. Plasmas **12**, 022307 (2005).

[24] R. Fitzpatrick, P.G. Watson, and F.L. Waelbroeck, Phys. Plasmas **12**, 082510 (2005).

[25] F.L. Waelbroeck, and R. Fitzpatrick, Phys. Rev. Lett. **78**, 1703 (1977).

[26] M. James, and H.R. Wilson, Plasma Phys. Control. Fusion **48**, 1647 (2006).

[27] M. James, and H.R. Wilson, Plasma Phys. Control. Fusion **52**, 075008 (2010).

[28] T.H. Stix, Phys. Fluids **16**, 1260 (1973).

[29] F.L. Hinton, and J.A. Robertson, Phys. Fluids **27**, 1243 (1984).

[30] A.I. Smolyakov, A. Hirose, E. Lazzaro, G.B. Re, and J.D. Callen, Phys. Plasmas **2**, 1581 (1995).

[31] E. Poli, A.G. Peeters, A. Bergmann, S. Günter, and S.D. Pinches, Phys. Rev. Lett. **88**, 075001 (2002).

[32] A. Bergmann, E. Poli, and A.G. Peeters, Phys. Plasmas **16**, 092507 (2009).

[33] H. Cai, D. Li, and J. Cao, Phys. Plasmas **22**, 102512 (2015).

[34] R. Fitzpatrick, and F.L Waelbroeck, Phys. Plasmas **16**, 072507 (2009).

[35] R. Fitzpatrick, and F.L. Waelbroeck, Phys. Plasmas **17**, 062503 (2010).

[36] R. Fitzpatrick, and F.L. Waelbroeck, Phys. Plasmas **19**, 112501 (2012).

[37] R.D. Hazeltine, M. Kotscheneuther, and P.J. Morrison, Phys. Fluids **28**, 2466 (1985).

[38] K.C. Shaing, Phys. Rev. Lett. **87**, 245003 (2001).

[39] K.C. Shaing, Phys. Plasmas **10**, 1443 (2003).

[40] A. Furuya, M. Yagi, and S.-I. Itoh, J. Phys. Soc. Jpn. **72**, 313 (2003).

[41] M. Kotschenreuther, R.D. Hazeltine, and P.J. Morrison, Phys. Fluids **28**, 294 (1985).

[42] Y.B. Kim, P.H. Diamond, and R.J. Groebner, Phys. Fluids B **3**, 2050 (1991).

[43] A.G. Peeters, Plasma Phys. Control. Fusion **42**, B231 (2000).

[44] R. Fitzpatrick, Nucl. Fusion **33**, 1049 (1993).

[45] P.H. Rutherford, *Basic Physical Processes of Toroidal Fusion Plasmas*, Proceedings of Course and Workshop, Varenna, 1985. (Commission of the European Communities, 1986). Vol. 2, p. 531.

[46] R.J. La Haye, C.C. Petty, E.J. Strait, F.L. Waelbroeck, and H.R. Wilson, Phys. Plasmas **10**, 3644 (2003).

[47] J.W. Connor, F.L. Waelbroeck, and H.R. Wilson, Phys. Plasmas **8**, 2835 (2001).

[48] M. Abramowitz, and I.A. Stegun (eds.), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables.* (Dover, 1965). Ch. 17.

[49] K. Imada, and H.R. Wilson, Plasma Phys. Control. Fusion **51**, 105010 (2009).

[50] K. Imada, and H.R. Wilson, Phys. Plasmas **19**, 032120 (2012).

[51] A.I. Smolyakov, and E. Lazzaro, Phys. Plasmas **11**, 4353 (2004).

Appendix A: Useful Integrals

Let $k = [(1+\Omega)/2]^{1/2}$. Then

$$\mathcal{A}(k) \equiv 2\,k\,\langle 1\rangle = \begin{cases} (2/\pi)\,k\,K(k) & 0 \leq k \leq 1 \\ (2/\pi)\,K(1/k) & k > 1 \end{cases}, \tag{A1}$$

and

$$\mathcal{B}(k) \equiv \langle |X|\rangle = \begin{cases} (2/\pi)\,\sin^{-1}(k) & 0 \leq k \leq 1 \\ 1 & k > 1 \end{cases}, \tag{A2}$$

and

$$\mathcal{C}(k) \equiv \frac{\langle X^{2}\rangle}{2\,k} = \begin{cases} (2/\pi)\,[E(k) + (k^{2}-1)\,K(k)]/k & 0 \leq k \leq 1 \\ (2/\pi)\,E(1/k) & k > 1 \end{cases}, \tag{A3}$$

and

$$\mathcal{D}(k) \equiv \frac{\langle |X|^{3}\rangle}{4\,k^{2}} = \begin{cases} (2/\pi)\,\sin^{-1}(k)\,[1 - 1/(2\,k^{2})] & 0 \leq k \leq 1 \\ 1 - 1/(2\,k^{2}) & k > 1 \end{cases}, \tag{A4}$$

and

$$\mathcal{E}(k) \equiv \frac{\langle X^{4}\rangle}{8\,k^{3}} = \begin{cases} (2/3\pi)\,[2\,(2 - 1/k^{2})\,E(k) + (3\,k^{2} - 5 + 2/k^{2})\,K(k)]\,/k & 0 \leq k \leq 1 \\ (2/3\pi)\,[2\,(2 - 1/k^{2})\,E(1/k) - (1 - 1/k^{2})\,K(1/k)] & k > 1 \end{cases}. \tag{A5}$$

Here,

$$E(k) = \int_0^{\pi/2} (1 - k^{2}\,\sin^2 u)^{1/2}\,du, \tag{A6}$$

$$K(k) = \int_0^{\pi/2} (1 - k^{2}\,\sin^2 u)^{-1/2}\,du \tag{A7}$$

are standard complete elliptic integrals.[48]

The following integrals are useful:

$$I_1 \equiv \int_0^\infty \frac{4\,[(2\,k^2-1)\,\mathcal{A}-2\,k^2\,\mathcal{C}]^2}{\mathcal{A}}\,dk = 0.823, \tag{A8}$$

$$I_2 \equiv \frac{2\pi}{3} - \int_1^\infty \frac{4}{\mathcal{C}}\left(\frac{\mathcal{E}\,\mathcal{A}}{\mathcal{C}^2}-1\right)dk = 1.38, \tag{A9}$$

$$I_3 \equiv \int_1^\infty 16\left(\frac{\mathcal{D}}{\mathcal{C}}-\frac{1}{\mathcal{A}}\right)k^2\,dk = 1.58, \tag{A10}$$

$$I_4 \equiv \int_1^\infty \frac{8}{\mathcal{C}}\left(1-\frac{1}{\mathcal{A}\,\mathcal{C}}\right)dk = 0.357, \tag{A11}$$

$$I_5 \equiv \frac{4}{2^{1/4}}\int_0^\infty \frac{dx}{1+x^4} = 2^{1/4}\,\pi = 3.74. \tag{A12}$$

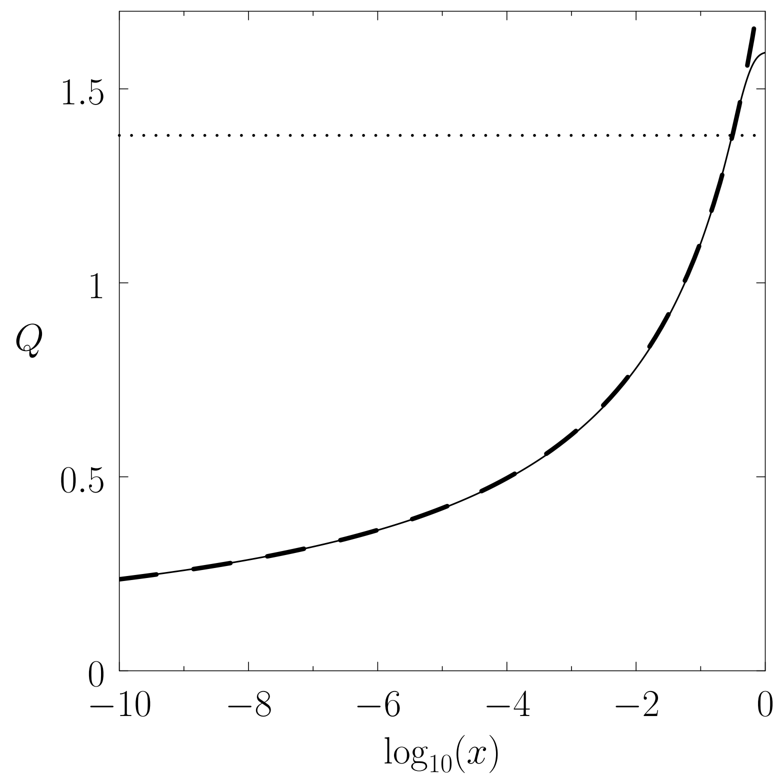

FIG. 1. The solid curve shows the separatrix boundary layer response function, $Q(x)$. The dashed curve shows the analytic approximation $Q(x) \simeq 6.2/\ln(16/x) - 3.0/\ln^2(16/x)$. The dotted line corresponds to $Q = 1.38$.

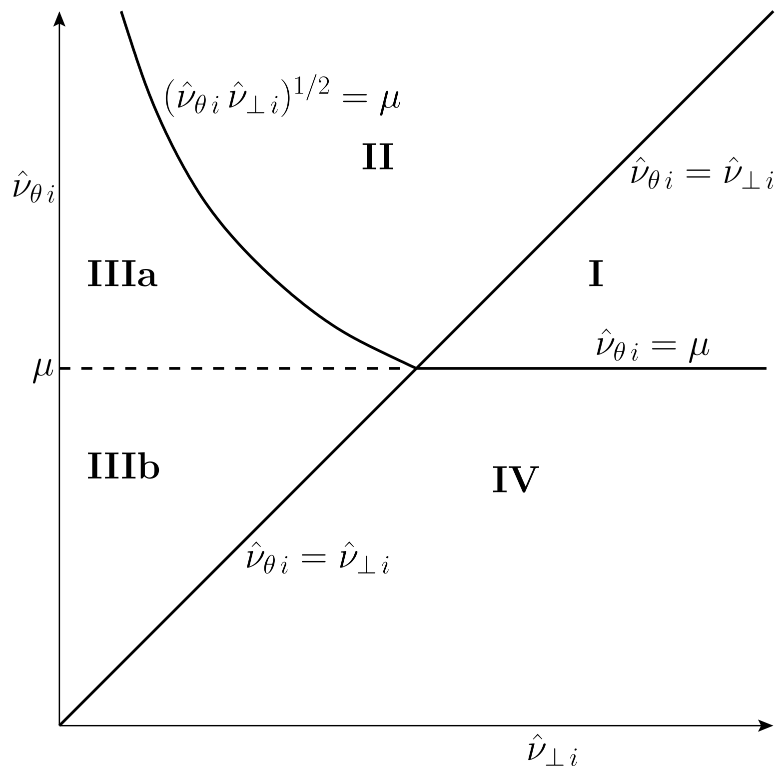

FIG. 2. Extends of various weak neoclassical ion poloidal flow damping island solution regimes in the $\hat{\nu}_{\perp\imath}$–$\hat{\nu}_{\theta\,i}$ plane.

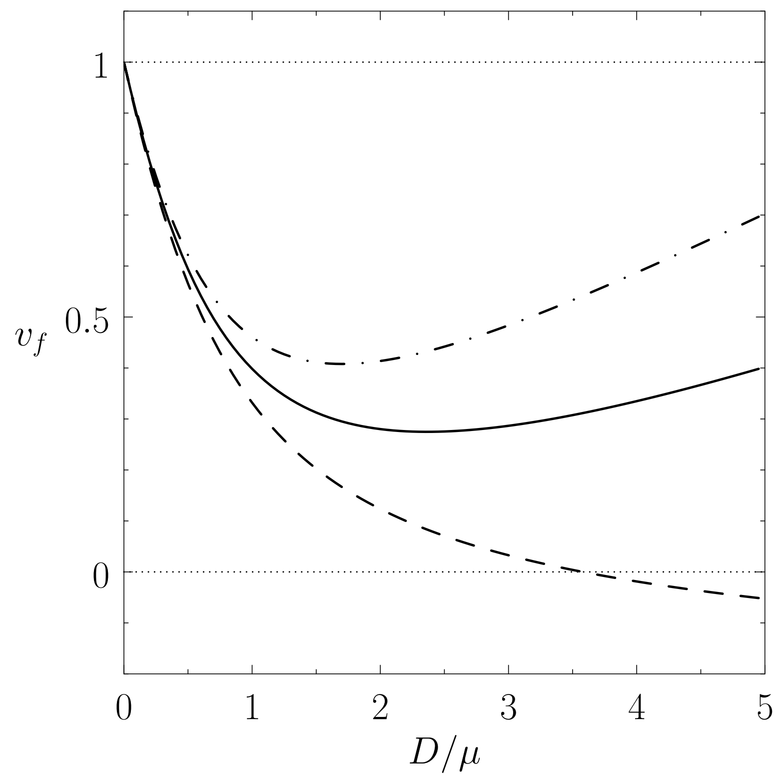

FIG. 3. The island phase-velocity parameter, v_f, calculated as a function of the perpendicular diffusivity ratio, D/μ, in the weak neoclassical ion perpendicular flow damping limit of the strong neoclassical ion poloidal flow damping regime. The dashed, solid, and dash-dotted curves correspond to $\Lambda = 0.5$, 0.0, and -0.5, respectively. All curves are calculated with $\hat{\tau} = 1$ and $\rho_{\theta\,i}/w = 1\times 10^{-3}$.

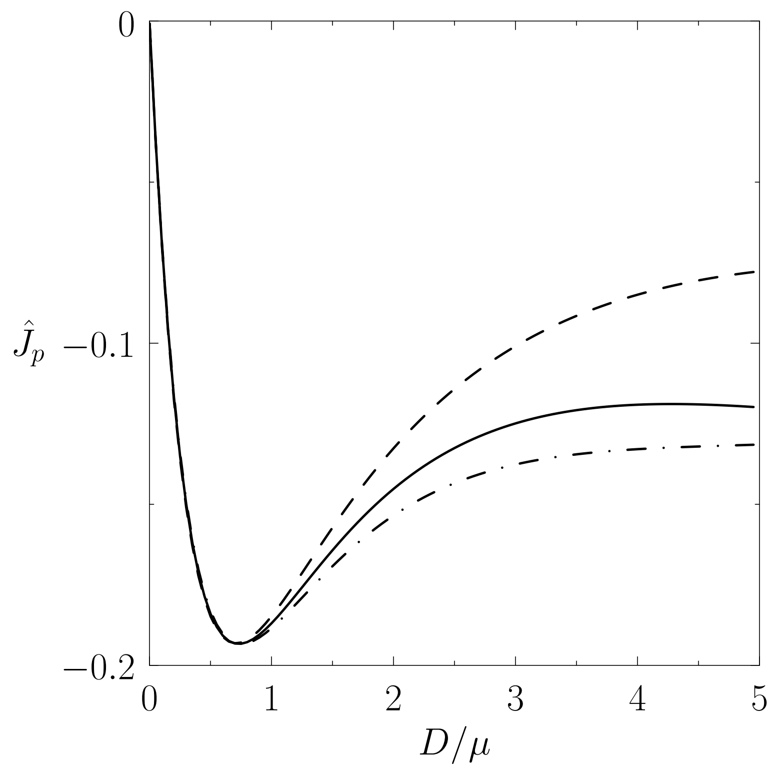

FIG. 4. The normalized ion polarization current integral, $\hat{J}_p$, calculated as a function of the perpendicular diffusivity ratio, D/μ, in the weak neoclassical ion perpendicular flow damping limit of the strong neoclassical ion poloidal flow damping regime. The dashed, solid, and dash-dotted curves correspond to $\Lambda = 0.5$, 0.0, and -0.5, respectively. All curves are calculated with $\hat{\tau} = 1$ and $\rho_{\theta\, i}/w = 10^{-3}$.

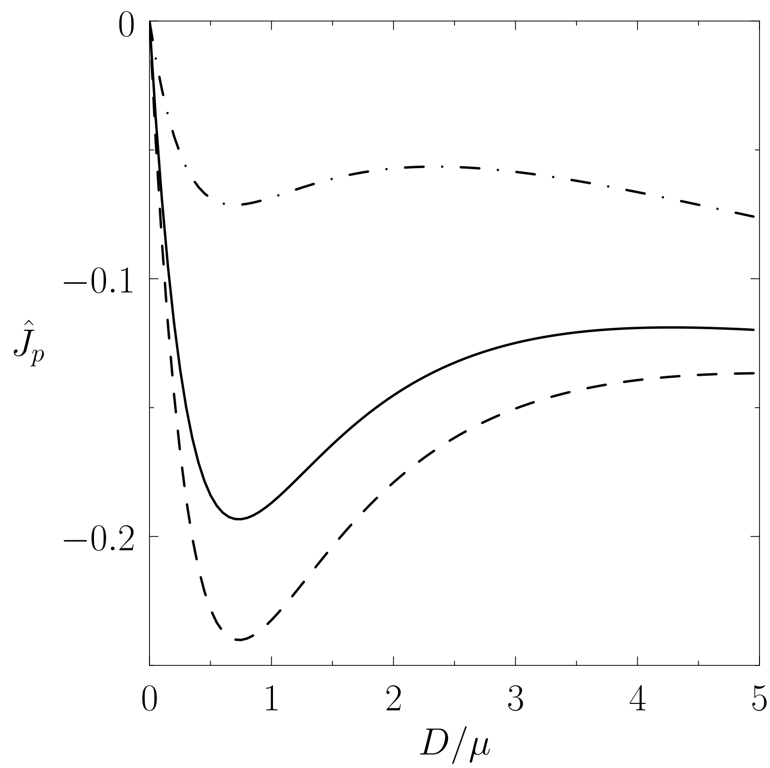

FIG. 5. The normalized ion polarization current integral, $\hat{J}_p$, calculated as a function of the perpendicular diffusivity ratio, D/μ, in the weak neoclassical ion perpendicular flow damping limit of the strong neoclassical ion poloidal flow damping regime. The dashed, solid, and dash-dotted curves correspond to $\rho_{\theta\, i}/w = 10^{-5}$, 10^{-3}, and 10^{-1}, respectively. All curves are calculated with $\hat{\tau} = 1$ and $\Lambda = 0$.

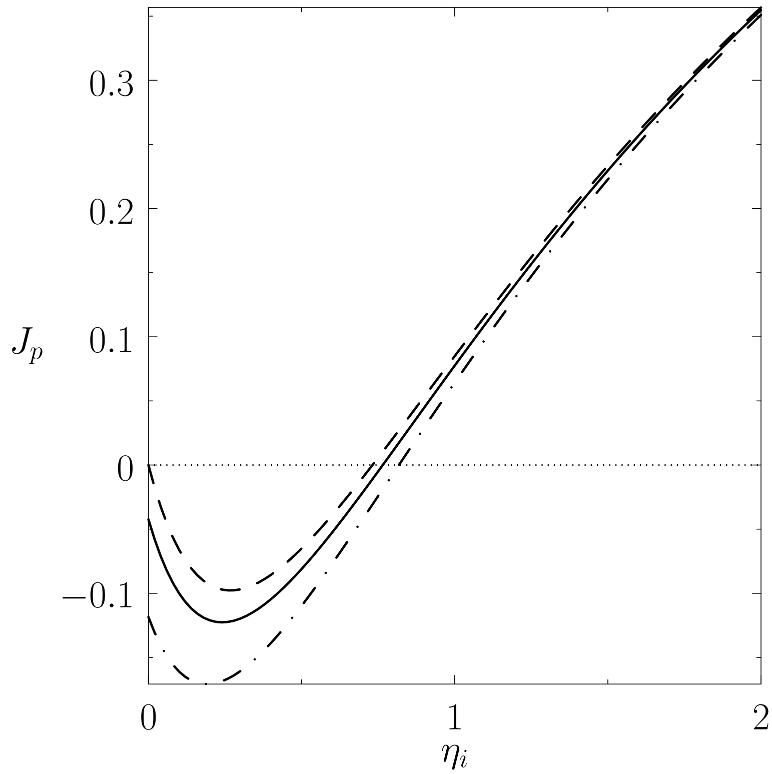

FIG. 6. The ion polarization current integral, J_p, calculated as a function of the ion temperature gradient parameter, η_i, for a locked island in the strong neoclassical ion perpendicular flow damping limit of the strong neoclassical ion poloidal flow damping regime. The dashed, solid, and dash-dotted curves correspond to $\delta_\perp/w = 10^{-3}$, 10^{-2}, and 10^{-1}, respectively. All curves are calculated with $\hat{\tau} = 1$, $D/\mu = 1$, and $\rho_{\theta\, i}/w = 10^{-3}$.

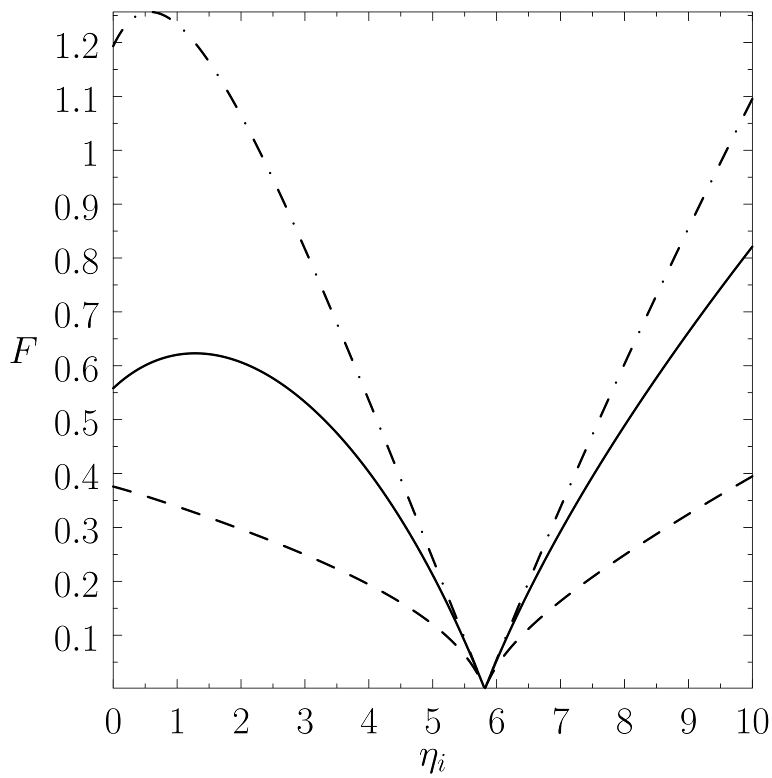

FIG. 7. The threshold neoclassical island width function, F, calculated as a function of η_i. The dashed, solid, and dash-dotted curves correspond to $\hat{\nu}_i = 0.1$, 1.0, and 10.0, respectively. The other calculation parameters are $\eta_e = \eta_i$, $T_e/T_i = 1$, $D/\mu = 1$, $L_q/L_c = 0$, $\epsilon_s = 0.1$